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SYM Description of PP-wave String Interactions: Singlet Sector and Arbitrary Impurities

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Abstract

We study string interactions among string states with arbitrary impurities in the Type IIB plane wave background using string field theory. We reproduce all string amplitudes from gauge theory by computing matrix elements of the dilatation operator in a previously proposed basis of states. A direct correspondence is found between the string field theory and gauge theory Feynman diagrams.

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1. Introduction

The solvability [1][2] of Type IIB string theory on its maximally supersymmetric plane wave geometry [3] has allowed BMN [4] to represent the *free* string spectrum of excitations around this background in terms of operators of $\mathcal{N} = 4$ SYM. Further checks of this identification were performed in [5] and culminated in [6], where the exact free spectrum of the string Hamiltonian was derived from gauge theory considerations for a class of string states.

The formulation of the duality between *interacting* string theory in the plane wave background and gauge theory has recently been formulated and tested in [7][8](see also [9] for a complementary description using the string bit model [10][11][12]). The basic idea is to extend the classical identification derived from the Penrose limit between the free string theory Hamiltonian H_2 and the gauge theory dilatation operator[1] Δ

$$\frac{1}{\mu}H_2 = \Delta - J, \tag{1.1}$$

to the full interacting theory. In the interacting theory the free Hamiltonian gets replaced by the interacting Hamiltonian $H = H_2 + g_2 H_3 + \ldots$, where g_2 is the string coupling constant, and the holographic map proposed in [7][8] reads

$$\frac{1}{\mu}H = \Delta - J. \tag{1.2}$$

In [9][8][7] a basis[2] of operators in $\mathcal{N} = 4$ SYM was found such that the $\mathcal{O}(g_2)$ matrix elements of the string Hamiltonian were reproduced using (1.2) from gauge theory computations, which were initiated in [13][14][15][16]. The analysis in [9][8][7] was restricted to string states with two different scalar impurities along an $\mathbf{R}^4$ plane in the transverse $\mathbf{R}^8$ directions of the plane wave. For previous work on string interactions in the plane wave background, see [14][17]-[32].

In this paper we compute the $\mathcal{O}(g_2)$ and $\mathcal{O}(g_2^2)$ Hamiltonian matrix elements for string states with two *identical* scalar impurities along $\mathbf{R}^4$ and reproduce them from gauge theory computations. We find that the matrix elements of the dilatation operator in the basis described in [8] exactly reproduces the string theory answer. When considering string states with identical impurities we find that there are new classes of Feynman diagrams

[1] J is the generator of a $U(1) \in SU(4)_R$ subgroup of the R-symmetry group of $\mathcal{N} = 4$ SYM.

[2] In the next section we will briefly review how to find the correct basis of gauge theory states.

that contribute to the string theory and gauge theory computations. In this work we find a direct connection between the Feynman diagrams that appear in the string calculation and the Feynman diagrams that contribute to the gauge theory matrix elements. Roughly, the action of the prefactor in string field theory is captured by the interaction vertex in gauge theory while the Neumann matrices are captured by the sum over all free contractions in gauge theory. This correspondence could be an important step in deriving the duality. We then compute the $\mathcal{O}(g_2)$ Hamiltonian matrix elements for string states with an *arbitrary* number of impurities along $\mathbf{R}^4$ and exactly reproduce them using gauge theory using (1.2) and the basis of states in [8], after identifying gauge theory Feynman diagrams with corresponding diagrams in string theory. These results give strong supporting evidence of the holographic map (1.2) and of the basis of gauge theory states proposed in [8] as a dual description of string states.

The plan of the rest of the paper is as follows. In section 2 we review the holographic map proposed in [7][8] and review the basis of states introduced in [8] in which to compute gauge theory quantities. In section 3 we consider the string states and gauge theory operators with two identical scalar impurities. We perform computations up to $\mathcal{O}(g_2^2)$ of the string Hamiltonian matrix elements, emphasizing the extra diagrams that contribute beyond those that appear when considering string states with two different scalar impurities. Using the basis change proposed in [8] we exactly reproduce the string theory results from a gauge theory analysis. In section 4 we show equivalence between string theory and gauge theory computations for arbitrary string states by identifying string theory Feynman diagrams with gauge theory Feynman diagrams. We conclude in section 5. Appendix I, which is outside the main focus of the paper, contains the $\mathcal{O}(g_2)$ calculation of a two-impurity p-string state transition into a $p+1$-string state. We find precise agreement with the gauge theory calculation in [33] once we change to the basis in [8]. The rest of the appendices summarize the calculations performed throughout the paper.

2. Review

As mentioned in the introduction, the proposal for the holographic map between string theory in the plane wave and $\mathcal{N}=4$ SYM is (1.2). This means, as anticipated by Verlinde [10], that all the information is encoded in the matrix of two point functions of BMN operators

$$|x|^{2\Delta_0}\langle O_A \bar{O}_B\rangle = G_{AB} + \Gamma_{AB}\ln(x^2\Lambda^2)^{-1}, \tag{2.1}$$

where G_{AB} is the inner product metric and Γ_{AB} is the matrix of anomalous dimensions. The proposal (1.2) requires the eigenvalues of H and $\mu(\Delta - J)$ to be the same. However, as emphasized in [7][9][8], comparison of matrix elements of these operators can be achieved in a suitable basis. The basic principle is to orthonormalize the gauge theory Hilbert space inner product G_{AB} order by order in g_2, which captures operator mixing[3] between BMN operators with different number of traces. By orthonormalizing, the gauge theory inner product coincides with the string theory Fock space inner product. The precise mapping between string theory Fock space states $|s_A\rangle$ and gauge theory orthonormal states $|\tilde{O}_A\rangle$ is given by

$$|s_A\rangle \rightarrow |\tilde{O}_A\rangle = U_{AB}|O_B\rangle, \text{ with } UGU^\dagger = \mathbf{1}, \text{ and } \langle s_A|s_B\rangle = \langle \tilde{O}_A|\tilde{O}_B\rangle = \delta_{AB}, \qquad (2.2)$$

where $|O_A\rangle$ are states created by BMN operators. Then, one can compute the matrix elements of the dilatation operator in the orthonormal basis and compare with string theory Hamiltonian matrix elements[4]

$$\frac{1}{\mu}\langle s_A|H|s_B\rangle = \langle \tilde{O}_A|(\Delta - J)\tilde{O}_B\rangle = (U\left(\left[\Delta^0 - J\right]G + \Gamma\right)U^\dagger)_{AB} = n\delta_{AB} + \tilde{\Gamma}_{AB}, \qquad (2.3)$$

where Γ is the matrix of anomalous dimensions of BMN operators and n is the number of impurities.

The change of basis U is, however, not unique. In [9][8][7] a basis was found for which the string theory matrix elements of string states with two different impurities was reproduced from gauge theory using (2.3). As emphasized in [8] the change of basis is the unique one which orthonormalizes the gauge theory inner product and leads to the matrix U being *real and symmetric*. We will show that this change of basis is universal by reproducing the string theory Hamiltonian matrix elements for arbitrary string states via matrix elements of the dilatation operator in the universal basis[5]. In this basis the matrix

[3] The relevance of mixing in the duality was first pointed out in [34].

[4] As in our previous paper, we will omit the trivial factor on the right hand side of (2.3) proportional to the classical dimension Δ^0 from now on.

[5] In our previous work [8], and in the rest of the paper we will refer to this basis as the string field theory basis.

of anomalous dimensions was evaluated in [8] and are given in terms of the BMN inner product metric G and matrix of anomalous dimensions Γ as

$$\begin{aligned}
&\tilde{\Gamma}^{(0)} = \Gamma^{(0)}, \\
&\tilde{\Gamma}^{(1)} = \Gamma^{(1)} - \frac{1}{2}\{G^{(1)}, \Gamma^{(0)}\}, \\
&\tilde{\Gamma}^{(2)} = \Gamma^{(2)} - \frac{1}{2}\{G^{(2)}, \Gamma^{(0)}\} - \frac{1}{2}\{G^{(1)}, \Gamma^{(1)}\} + \frac{3}{8}\{\left(G^{(1)}\right)^2, \Gamma^{(0)}\} + \frac{1}{4}G^{(1)}\Gamma^{(0)}G^{(1)},
\end{aligned} \tag{2.4}$$

where $M^{(s)}$, with $M = \Gamma, G$ or $\tilde{\Gamma}$, is the g_2^s term in the expansion of $M = M^{(0)} + g_2 M^{(1)} + g_2^2 M^{(2)} + \ldots$.

3. Correspondence in two impurity singlet sector

In this section, we study string states and BMN operators with two real scalar impurities along the same direction in $\mathbf{R}^4$. Since $SO(4)$ is a symmetry, we can decompose two scalar impurity states into $\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9}$ irreducible representations of $SO(4)$, with two repeated impurities belonging to the singlet. We will consider states with two impurities in one direction $i \in \{1, 2, 3, 4\}$ instead of looking at the singlet state and later on extend the analysis to arbitrary number of impurities.

The single string states we will consider are given by (no sum over i):

$$\begin{aligned}
|ii, n\rangle &= \alpha_n^{i\dagger} \alpha_{-n}^{i\dagger} |\text{vac}\rangle, \\
|ii, 0\rangle &= \frac{1}{\sqrt{2}} \alpha_0^{i\dagger} \alpha_0^{i\dagger} |\text{vac}\rangle.
\end{aligned} \tag{3.1}$$

As shown by [35][15], the corresponding gauge theory operators when $g_2 = 0$ are given respectively by

$$\begin{aligned}
\mathcal{O}_{ii,n}^J &= \frac{1}{\sqrt{JN^{J+2}}} \left(\sum_{l=0}^{J} e^{2\pi i l n/J} \mathrm{Tr}\left(\phi_i Z^l \phi_i Z^{J-l}\right) - \mathrm{Tr}\left(\bar{Z} Z^{J+1}\right) \right), \\
\mathcal{O}_{ii,0}^J &= \frac{1}{\sqrt{2JN^{J+2}}} \left(\sum_{l=0}^{J} \mathrm{Tr}\left(\phi_i Z^l \phi_i Z^{J-l}\right) - \mathrm{Tr}\left(\bar{Z} Z^{J+1}\right) \right),
\end{aligned} \tag{3.2}$$

without summing over i. The extra contribution involving $\bar{Z}$ is crucial [35][15] for the existence of the BMN limit, where $N, J \to \infty$, with $g,\ g_2 = J^2/N$ and $\lambda' = g^2 N/J^2$ fixed and as we will see leads to interesting new effects.

The interaction term H_3 couples single string states to two-string states. These are given by

$$
\begin{aligned}
|ii,m,y\rangle\rangle &= \alpha_m^{i\dagger}\alpha_{-m}^{i\dagger}|\mathrm{vac},y\rangle \otimes |\mathrm{vac},1-y\rangle, \\
|ii,0,y\rangle\rangle &= \frac{1}{\sqrt{2}}\alpha_0^{i\dagger}\alpha_0^{i\dagger}|\mathrm{vac},y\rangle \otimes |\mathrm{vac},1-y\rangle, \\
|ii,y\rangle\rangle &= \alpha_0^{i\dagger}|\mathrm{vac},y\rangle \otimes \alpha_0^{i\dagger}|\mathrm{vac},1-y\rangle,
\end{aligned}
\tag{3.3}
$$

where $0 < y < 1$ is the fraction of the total momentum carried by the first string in the two-string state. These states are represented when $g_2 = 0$ by the following gauge theory operators

$$
\begin{aligned}
\mathcal{T}_{ii,m}^{J,y} &=: \mathcal{O}_{ii,m}^{y\cdot J} \cdot \mathcal{O}^{(1-y)\cdot J} :, \\
\mathcal{T}_{ii}^{J,y} &=: \mathcal{O}_i^{y\cdot J} \cdot \mathcal{O}_i^{(1-y)\cdot J} :,
\end{aligned}
\tag{3.4}
$$

where $y = J_1/J$ and $1-y = J_2/J$ and

$$
\begin{aligned}
\mathcal{O}^J &= \frac{1}{\sqrt{JN^J}}\mathrm{Tr}\left(Z^J\right), \\
\mathcal{O}_i^J &= \frac{1}{\sqrt{N^{J+1}}}\mathrm{Tr}\left(\phi_i Z^J\right).
\end{aligned}
\tag{3.5}
$$

We now proceed to describe string interactions among these states using string field theory and reproduce the results from a gauge theory analysis.

3.1. SFT computations

The proper way to describe string interactions in the light-cone gauge is by using light-cone string field theory. The Hamiltonian is given by $H = H_2 + g_2 H_3 + g_2^2 H_2' + \ldots$, where g_2 is the string coupling constant. H_3 is the leading interaction and couples an n-string state to an $(n\pm 1)$-string state and H_2' is a contact term. Following the flat space results in [36][37] the plane wave vertex H_3 has been studied in [38][39][40][41][42].

• *The $\mathcal{O}(g_2)$ Computation*

The properly normalized cubic interaction term in the case of purely bosonic excitations along $\mathbf{R}^4$ in the exponential (BMN) basis of oscillators is given by[6]

$$
\frac{1}{\mu}|H_3\rangle = -\frac{y(1-y)}{2}P|V\rangle,
\tag{3.6}
$$

[6] We take without loss of generality $\alpha' p_{(3)}^+ = -1, \alpha' p_{(1)}^+ = y$ and $\alpha' p_{(2)}^+ = 1-y$, where $0 < y < 1$. Therefore, $\lambda' = 1/\mu^2$. The large μ normalization was fixed in [9][8] by comparison with a field theory amplitude.

where $p^+_{(r)}$ is the length of string r and P is the prefactor

$$P = \sum_{r=1}^{3} \sum_{n=-\infty}^{\infty} \frac{\omega_{n(r)}}{\mu p^+_{(r)} \alpha'} \alpha^{i\dagger}_{n(r)} \alpha^i_{-n(r)}, \tag{3.7}$$

with $\omega_{n(r)} = \sqrt{(\mu p^+_{(r)} \alpha')^2 + n^2}$ and[7]

$$|V\rangle = \exp\Bigg(\frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n=-\infty}^{\infty} \alpha^{i\dagger}_{m(r)} \tilde{N}^{(rs)}_{mn} \alpha^{i\dagger}_{n(s)} \Bigg) |\text{vac}\rangle. \tag{3.8}$$

We now compute the matrix elements between single string and two-string states. It is convenient to introduce Feynman rules to evaluate these amplitudes, specially in later sections when we consider arbitrary impurities. They are given by:

$$\begin{aligned} (r,m) \;\text{———}\; (s,n) \quad &\Longleftrightarrow \quad \tilde{N}^{(rs)}_{m,n}, \\ (r,m) \;\text{—}\!\!\times\!\!\text{—}\; (s,n) \quad &\Longleftrightarrow \quad \left[\frac{\omega_{m(r)}}{\mu p^+_{(r)} \alpha'} + \frac{\omega_{n(s)}}{\mu p^+_{(s)} \alpha'} \right] \tilde{N}^{(rs)}_{m,-n}, \end{aligned} \tag{3.9}$$

where $r, s \in \{1, 2, 3\}$ label the string and m, n label the worldsheet momentum of the oscillator. Then, the Neumann matrix $\tilde{N}^{(rs)}_{m,n}$ is the propagator between oscillators $\alpha_{m(r)}$ and $\alpha_{n(s)}$. We can eliminate the prefactor P in (3.6) by sequentially commuting it through the external states oscillators, which has the effect of reversing the sign of the worldsheet momentum of the oscillator which P is acting on. After elimination of the prefactor, we are left with contractions between external states oscillators. The $\times$ symbol in the vertex (3.9) signifies the total effect of commuting the prefactor P in (3.6) through both oscillators and their contraction.

Using these Feynman rules and the following symmetry relations satisfied by the Neumann matrices

$$\tilde{N}^{(rs)}_{m,n} = \tilde{N}^{(sr)}_{n,m}, \qquad \tilde{N}^{(rs)}_{m,n} = \tilde{N}^{(rs)}_{-m,-n}, \tag{3.10}$$

[7] Here we omit the overall p^+ conservation factor, $|p^+_{(3)}| \delta(p^+_{(1)} + p^+_{(2)} + p^+_{(3)})$.

we can now evaluate any Hamiltonian matrix element using combinatorics of Feynman diagrams. In the case of two identical impurities, the amplitudes are given by:

$$
\begin{aligned}
\frac{1}{\mu}\langle ii,n|H_3|jj,m,y\rangle\rangle = &-\frac{y(1-y)}{2}\Bigg[\delta_{ij}4\tilde{N}^{(13)}_{m,n}\tilde{N}^{(13)}_{m,-n}\left(\frac{\omega_{m(1)}}{\mu y}-\frac{\omega_{n(3)}}{\mu}\right) \\
&+2\tilde{N}^{(33)}_{n,-n}\tilde{N}^{(11)}_{m,m}\frac{\omega_{m(1)}}{\mu y}-2\tilde{N}^{(33)}_{n,n}\tilde{N}^{(11)}_{m,-m}\frac{\omega_{n(3)}}{\mu}\Bigg], \\
\frac{1}{\mu}\langle ii,n|H_3|jj,y\rangle\rangle = &-\frac{y(1-y)}{2}\Bigg[\delta_{ij}4\tilde{N}^{(13)}_{0,n}\tilde{N}^{(23)}_{0,n}\left(1-\frac{\omega_{n(3)}}{\mu}\right) \\
&+2\tilde{N}^{(33)}_{n,-n}\tilde{N}^{(12)}_{0,0}-2\tilde{N}^{(33)}_{n,n}\tilde{N}^{(12)}_{0,0}\frac{\omega_{n(3)}}{\mu}\Bigg].
\end{aligned}
\tag{3.11}
$$

We note that there are Feynman diagrams in which the identical impurities in a given string are connected via Neumann matrices involving only that string. Such contributions are absent when considering strings with different impurities due to the $SO(8)$ invariance of the Neumann matrices. We can evaluate the expression in the large[8] μ limit, which corresponds to the perturbative gauge theory regime. Even though $\tilde{N}^{(11)}$, $\tilde{N}^{(12)}$ and $\tilde{N}^{(33)}$ are suppressed by $1/\mu$ as compared to $\tilde{N}^{(13)}$, the self-contraction contributions are of the same order as the contractions between different strings due to cancellations in the contribution of contractions between different strings. The large μ expressions are given by[9]

$$
\begin{aligned}
\frac{1}{\mu}\langle ii,n|H_3|jj,m,y\rangle\rangle &= \delta_{ij}\left(\tilde{\Gamma}^{(1)}_{n,my}+\tilde{\Gamma}^{(1)}_{-n,my}\right)-\frac{1}{2}\Gamma^{(1)}_{n,0y}, \\
\frac{1}{\mu}\langle ii,n|H_3|jj,y\rangle\rangle &= \delta_{ij}\left(\tilde{\Gamma}^{(1)}_{n,y}+\tilde{\Gamma}^{(1)}_{-n,y}\right)-\frac{1}{2}\Gamma^{(1)}_{n,y},
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
\tilde{\Gamma}^{(1)}_{n,my} &= \lambda'\frac{\sqrt{1-y}}{\sqrt{Jy}}\frac{\sin^2(\pi n y)}{2\pi^2}, \\
\tilde{\Gamma}^{(1)}_{n,y} &= -\lambda'\frac{1}{\sqrt{J}}\frac{\sin^2(\pi n y)}{2\pi^2}.
\end{aligned}
\tag{3.13}
$$

$\Gamma^{(1)}_{n,0y}$ and $\Gamma^{(1)}_{n,y}$ are defined in Appendix B and as we shall see have a direct gauge theory origin. The splitting of the first term in (3.12) into two identical contributions is convenient when comparing with the gauge theory analysis in the next subsection.

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[8] We summarize the large μ expansion of the Neumann matrices in the Appendix A.

[9] In light-cone string field theory the canonical normalization of states is the usual delta function normalization $\langle s'_A|s'_B\rangle = |p^+_A|\delta(p^+_A+p^+_B) = J_A\delta_{J_A,J_B}$, so that $|s'_A\rangle = \sqrt{J_A}|s_A\rangle$. Therefore, when comparing string field theory results with gauge theory results we have to take into account this normalization factor and the overall δ-function in (3.8), since gauge theory states have unit norm [43][14], so we divide the string theory answer (3.11) by $\sqrt{Jy(1-y)}$.

The first contribution in (3.12) is twice as large as compared to the answer one gets when considering string states with two different impurities [9][8]. The reason is that there are twice as many ways of contracting impurities among different strings. This is reproduced in the gauge theory computation because the scalar impurities have two ways of contracting when they are both the same. The last term in (3.12) are due to self-contractions and only appear when two impurities are repeated. In the gauge theory computation in next subsection these extra contractions are due to the extra diagrams that one gets when considering the operators (3.2)(3.4). The new contractions in string field theory correspond to gauge theory diagrams involving $\bar{Z}$ and diagrams coupling all four scalar impurities. In section 4 the connection between gauge theory diagrams and string field theory diagrams will be made explicit.

• *The* $\mathcal{O}(g_2^2)$ *Computation*

We now consider the $\mathcal{O}(g_2^2)$ matrix elements between single string states, that is, the contact term contribution. We will also reproduce this result from gauge theory considerations.

The single string contact term in the plane wave geometry has been recently analyzed in [44]. It is constructed from the plane wave dynamical supersymmetry generators via $H_2' = \{Q_3, \bar{Q}_3\}$, where Q_3 is the leading g_2 correction to the free supercharge. In [44] it was shown that by considering the contact term contribution for two different impurity string states the gauge theory results in the orthonormal basis of [9][8][7] could be reproduced if one truncated the intermediate states to the two impurity sector. We will perform a similar calculation for string states with two identical impurities using the same truncation and reproduce these results from gauge theory in the next subsection. Understanding more precisely why the truncation works is an important open problem.

The intermediate two impurity states that contribute are given by

$$\begin{aligned}
|j,m,y,1\rangle\rangle &= \alpha_m^{j\dagger}\frac{1}{\sqrt{2}}(b_m^{d\dagger} - ie(m)b_{-m}^{d\dagger})|\text{vac},y\rangle \otimes |\text{vac},1-y\rangle, \\
|j,0,y,1\rangle\rangle &= \alpha_0^{j\dagger} b_0^{d\dagger}|\text{vac},y\rangle \otimes |\text{vac},1-y\rangle, \\
|j,0,y,1\rangle\rangle' &= \alpha_0^{j\dagger}|\text{vac},y\rangle \otimes b_0^{d\dagger}|\text{vac},1-y\rangle,
\end{aligned} \tag{3.14}$$

and $|j,m,y,2\rangle\rangle(')$ defined by changing the string on which the operators act. The b oscillators are the fermionic oscillators. Using the expression in [41] for the supersymmetry charge Q we can calculate its matrix elements in the large μ limit[10](see Appendix C for details)

$$
\begin{aligned}
Q_{n,m(s)} &= \langle ii,n|Q_{\dot{a}}|j,m,y,s\rangle\rangle \\
&\simeq \sqrt{1+\mu\alpha k}\delta_{ij}u^i_{abc\dot{a}}\delta^{abcd}_{1234}\frac{Y_{m(s)}}{\sqrt{2}}\left[\tilde{F}^-_{(3)-n}\tilde{N}^{(s3)}_{m,n}+\tilde{F}^-_{(3)n}\tilde{N}^{(s3)}_{m,-n}\right],
\end{aligned}
\tag{3.15}
$$

for $s=1,2$. Therefore the $\mathcal{O}(g_2^2)$ Hamiltonian matrix element in the case of two impurities in the same direction is given by

$$
\langle ii,n|H_2'|jj,m\rangle = \delta_{ij}\int_0^1 \frac{dy}{y(1-y)}\sum_{s=1}^{2}\sum_{l=-\infty}^{\infty} Q_{n,l(s)}Q^*_{m,l(s)}. \tag{3.16}
$$

Performing the relevant sums and integral one arrives at the final result(see Appendix D):

$$
\langle ii,n|H_2'|jj,m\rangle = \delta_{ij}\frac{1}{16\pi^2}(B_{n,m}+B_{n,-m}). \tag{3.17}
$$

The result in (3.17) has an extra term as compared to the calculation for two different impurities, which is identical to the first one except for the sign of the worldsheet momentum.

In this subsection we have calculated the Hamiltonian matrix elements using string field theory up to $\mathcal{O}(g_2^2)$. We now turn to the gauge theory analysis.

3.2. Gauge theory computations

The BMN operators with two identical scalar impurities (3.2)(3.4) are insensitive to the sign of the worldsheet momentum since $\mathcal{O}^J_{ii,n}=\mathcal{O}^J_{ii,-n}$ and $\mathcal{T}^{J,y}_{ii,m}=\mathcal{T}^{J,y}_{ii,-m}$, so we will consider without loss of generality $n,m\geq 0$. Moreover, the BPS double trace operator $\mathcal{T}^{J,y}_{ii}$ is invariant under $y\to 1-y$, so we can restrict to $0<y\leq 1/2$.

As explained in section 2, in order to compute string interactions from gauge theory we must compute the matrix of two point functions of BMN operators $\mathcal{O}^J_{ii,n}$, $\mathcal{T}^{J,y}_{ii,p}$ and $\mathcal{T}^{J,y}_{ii}$. The relevant inner product metric and matrix of anomalous dimensions can be extracted from [15]. They are given by[11]:

$$
\begin{aligned}
G = \mathbf{1} &+ g_2\,\delta_{ij}\begin{pmatrix} 0 & C_{n,qz}+C_{-n,qz} & 2C_{n,z} \\ C_{py,m}+C_{py,-m} & 0 & 0 \\ 2C_{y,m} & 0 & 0 \end{pmatrix} \\
&+ g_2^2\,\delta_{ij}\begin{pmatrix} M^1_{n,m}+M^1_{n,-m} & 0 & 0 \\ 0 & \langle ? \rangle & \langle ? \rangle \\ 0 & \langle ? \rangle & \langle ? \rangle \end{pmatrix},
\end{aligned}
\tag{3.18}
$$

[10] The zero mode contribution vanishes in the large μ regime.

[11] We have summarized in the Appendix B the explicit expressions for the matrix elements.

and

$$\Gamma = \delta_{ij}\begin{pmatrix} \lambda' n^2\delta_{nm} & 0 & 0 \\ 0 & \lambda'\frac{p^2}{y^2}\delta_{p,q}\delta_{y,z} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$+ g_2 \begin{pmatrix} 0 & \delta_{ij}\left(\Gamma^{(1)}_{n,qz}+\Gamma^{(1)}_{-n,qz}\right)-\frac{1}{2}\Gamma^{(1)}_{n,0z} & 2\delta_{ij}\Gamma^{(1)}_{n,z}-\frac{1}{2}\Gamma^{(1)}_{n,z} \\ \delta_{ij}\left(\Gamma^{(1)}_{py,m}+\Gamma^{(1)}_{py,-m}\right)-\frac{1}{2}\Gamma^{(1)}_{0y,m} & 0 & 0 \\ 2\delta_{ij}\Gamma^{(1)}_{y,m}-\frac{1}{2}\Gamma^{(1)}_{y,m} & 0 & 0 \end{pmatrix}$$
$$+ g_2^2 \begin{pmatrix} \delta_{ij}\left(\Gamma^{(2)}_{n,m}+\Gamma^{(2)}_{n,-m}\right)-\frac{1}{16\pi^2}\mathcal{D}^1_{n,m} & 0 & 0 \\ 0 & \langle ? \rangle & \langle ? \rangle \\ 0 & \langle ? \rangle & \langle ? \rangle \end{pmatrix}, \tag{3.19}$$

where $\langle ? \rangle$ denotes matrix elements that have not yet been computed. We note that whenever the worldsheet momentum index in (3.18)(3.19) vanishes, that we must divide the matrix element by $\sqrt{2}$. Likewise when both operators have vanishing momentum we must divide that matrix element by 2. These extra factors arise from our normalization of the operators in (3.2)(3.4) which differ from those in [15]. In this way we get an orthonormal inner product for $n, m \geq 0$.

The inner product metric can be computed in the free theory while the matrix of anomalous dimensions comes with a power of λ' from evaluating one loop graphs. In the free theory the $\bar{Z}$ portion of the gauge theory operators (3.2)(3.4) does not couple to the terms in (3.2)(3.4) without the $\bar{Z}$. Moreover, the diagrams involving only $\bar{Z}$ are suppressed by a power of $1/J$ with respect to the leading contribution, which only involves the part of the operator with the two impurities(terms without $\bar{Z}$). Therefore, in the computation of the mixing matrix the extra term in the operators (3.2)(3.4) does not contribute in the BMN limit, so that at any order in g_2 the inner product metric can be calculated neglecting the $\bar{Z}$ term. It then follows that there are twice as many contributions in the inner product of (3.2)(3.4) as compared to the case of two different impurities. This is easy to understand since there are now twice as many ways of contracting the impurities and they come with the opposite sign of the worldsheet momentum. An analogous phenomena occurs when extending the analysis to arbitrary number of impurities.

The matrix of anomalous dimensions also has twice as many contributions of the type appearing for different impurities. These gauge theory Feynman diagrams can be identified in the string field theory calculation with contractions involving impurities living

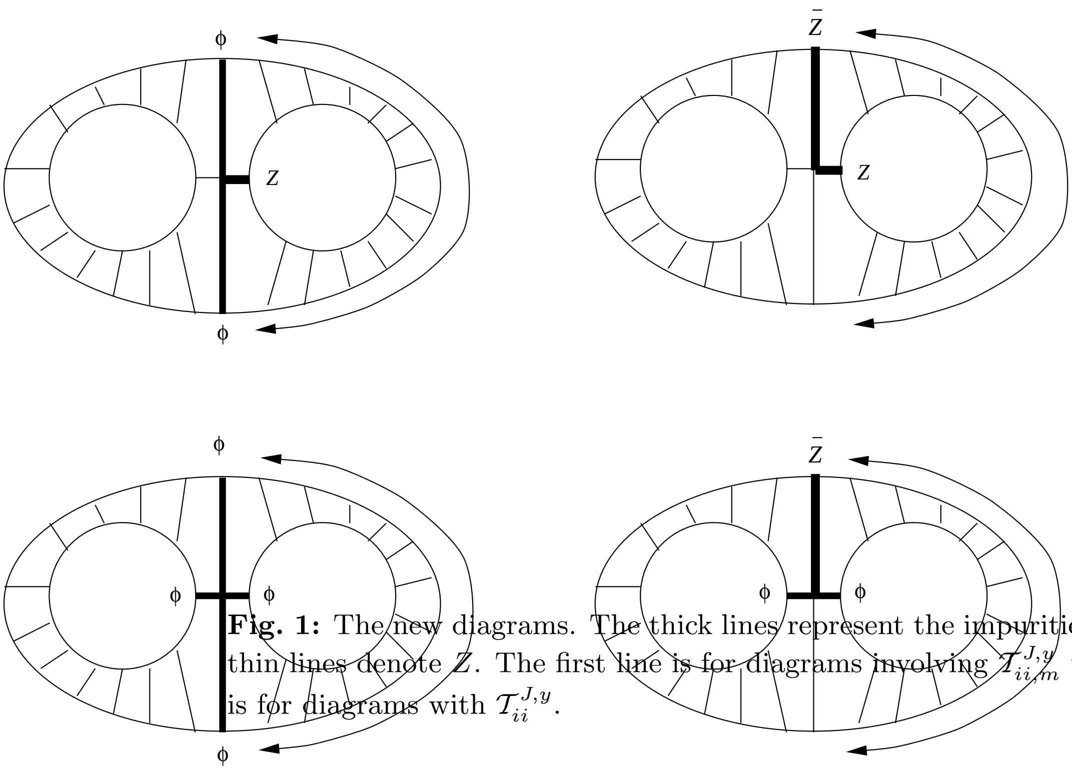

Fig. 1: The new diagrams. The thick lines represent the impurities or $\bar{Z}$ while the thin lines denote Z. The first line is for diagrams involving $\mathcal{T}_{ii,m}^{J,y}$ while the second is for diagrams with $\mathcal{T}_{ii}^{J,y}$.

in different strings. However, there is an extra term arising from vertices involving $\bar{Z}$ in (3.2)(3.4) and the coupling of all scalar impurities[12] (see Fig. 1).

These extra Feynman diagrams can be identified in the string field theory calculation with contractions of impurities living on the same string as can be inferred by looking at (3.12).

[12] The quartic scalar coupling denotes the effective interaction after taking into account self-energy and gluon exchange diagrams[15].

We can now test the holographic correspondence (2.3). Using the formula for the matrix of anomalous dimensions in the orthonormal basis in terms of G and Γ we find:

$$\begin{aligned} \tilde{\Gamma}^{(1)}_{ii;n,jj;my} &= \delta_{ij}\left(\tilde{\Gamma}^{(1)}_{n,my} + \tilde{\Gamma}^{(1)}_{-n,my}\right) - \frac{1}{2}\Gamma^{(1)}_{n,0y}, \\ \tilde{\Gamma}^{(1)}_{ii;n,jj;y} &= \delta_{ij}\left(\tilde{\Gamma}^{(1)}_{n,y} + \tilde{\Gamma}^{(1)}_{-n,y}\right) - \frac{1}{2}\Gamma^{(1)}_{n,y}, \end{aligned} \tag{3.20}$$

where terms with $\tilde{\Gamma}$ on the right hand side come from the usual Feynman diagrams present also for two different impurities and the last term comes from new diagrams only present when two impurities are the same. By comparing with the string field theory calculation (3.12) we find precise agreement.

We now proceed to computing the matrix elements of the mostly single trace operators[13] to order g_2^2. Using (2.4) we find after some computation[14]

$$\tilde{\Gamma}^{(2)}_{ii;n,jj;m} = \delta_{ij}\left(\tilde{\Gamma}^{(2)}_{n,m} + \tilde{\Gamma}^{(2)}_{n,-m}\right) + \delta\tilde{\Gamma}^{(2)}_{ii;n,jj;m}, \tag{3.21}$$

where

$$\tilde{\Gamma}^{(2)}_{n,m} = \frac{1}{16\pi^2} B_{n,m}, \tag{3.22}$$

is the result obtained for different impurity operators [8] and $\delta\tilde{\Gamma}^{(2)}_{ii;n,jj;m}$ are the new contributions only arising for identical impurity operators. They are given by[15]

$$\delta\tilde{\Gamma}^{(2)}_{ii;n,jj;m} = \delta\Gamma^{(2)}_{ii;n,jj;m} - \frac{1}{2}\{G^{(1)}, \delta\Gamma^{(1)}\}_{ii;n,jj;m}, \tag{3.23}$$

since as explained above only the matrix of anomalous dimensions receives genuine new contributions while the inner product contributions have the same form as in the case of different impurities. From (3.19) we read

$$\begin{aligned} \delta\Gamma^{(2)}_{ii;n,jj;m} &= -\frac{1}{16\pi^2}\mathcal{D}^1_{n,m}, \\ \delta\Gamma^{(1)}_{ii;n,jj;my} = \delta\Gamma^{(1)}_{ii;my,jj;n} &= -\frac{1}{2}\Gamma^{(1)}_{n,0y}, \\ \delta\Gamma^{(1)}_{ii;n,jj;y} = \delta\Gamma^{(1)}_{ii;y,jj;n} &= -\frac{1}{2}\Gamma^{(1)}_{n,y}. \end{aligned} \tag{3.24}$$

[13] In order to compute the matrix elements of the mostly double trace operators to this order, we would need to know the expressions for $\langle ? \rangle$.

[14] For the detailed computation, see Appendix E.

[15] We use the notation δA for the new contributions to A due to having identical impurities.

After some computation one finds (see Appendix E for details)

$$\{G^{(1)}, \delta\Gamma^{(1)}\}_{ii;n,jj;m} = -\frac{1}{8\pi^2}\mathcal{D}^1_{n,m}, \tag{3.25}$$

giving us the simple result:

$$\delta\tilde{\Gamma}^{(2)}_{ii;n,jj;m} = 0. \tag{3.26}$$

Hence, the final expression is

$$\tilde{\Gamma}^{(2)}_{ii;n,jj;m} = \delta_{ij}\frac{1}{16\pi^2}(B_{n,m} + B_{n,-m}), \tag{3.27}$$

which exactly matches the $\mathcal{O}(g_2^2)$ contact term contribution in the string field theory calculation (3.17).

We now turn to the analysis of arbitrary string states.

4. Generalization to arbitrary impurities

Thus far we have analyzed the correspondence for string states with two impurities. In this section we construct a proof that shows the equivalence between the string theory and gauge theory computations for an arbitrary number of impurities. The idea is to find a direct link between the Feynman diagrams of string theory and the Feynman diagrams of gauge theory, so that the equality between string theory and gauge theory for arbitrary states follows diagram by diagram. We first outline the strategy of the proof and then give the explicit details of the string theory and gauge theory computation.

Let's first consider which diagrams in string theory contribute to leading order in the $1/\mu$ expansion, which is of $\mathcal{O}(1/\mu^2)$. These diagrams will have corresponding contributions in the one loop – which is of $\mathcal{O}(\lambda')$ – gauge theory computation. We consider matrix elements between single string states and two-string states with n impurities each, that is impurity preserving[16] processes. The impurities can be distributed at will among the four directions in $\mathbf{R}^4$.

As explained in section 3, in order to compute $\mathcal{O}(g_2)$ Hamiltonian matrix elements, we must commute the prefactor (3.7) of the cubic vertex (3.6) through all the impurities. These gives us a sum of $2n$ terms with $2n$ oscillators each in which the sign of the worldsheet momentum of one of the oscillators is reserved. Each term now can be calculated using

[16] Impurity non-preserving processes are inherently non-perturbative [14].

the Feynman rules (3.9). Each diagram is multiplied by the frequency of the oscillator whose worldsheet momentum is reversed when commuting through the prefactor. Now, given the $SO(8)$ invariance of the string field theory vertex (3.8), the oscillators in different directions in $\mathbf{R}^4$ completely decouple, so we can concentrate on the case in which all the impurities are in one direction. The final answer for arbitrary string states is just the product of the contribution along each of the $\mathbf{R}^4$ directions.

We can now classify Feynman diagrams in terms of the number of self-contractions (propagators) in the single string state, that is the number of $\tilde{N}^{(33)}$'s. It is clear that to $\mathcal{O}(1/\mu^2)$ there can be *at most* one self-contraction. Since we are looking at impurity preserving processes, a self-contraction $\tilde{N}^{(33)}$ always is accompanied by a self-contraction in the two-string state of the type $\tilde{N}^{(rs)}$, where r, s is either 1 or 2. Since $\tilde{N}^{(33)}$ and $\tilde{N}^{(rs)}$ are of $\mathcal{O}(1/\mu)$, we can have at most one self-contraction to leading order in the $1/\mu$ expansion. This simple observation greatly diminishes the Feynman diagrams that need to be considered. We now study the two possibilities.

Let us consider first the case in which there are no self-contractions. In this case all impurities in the single string are contracted with impurities of the two-string state, so the result is the product of Neumann matrices of the type $\tilde{N}^{(r3)}, (r = 1, 2)$, where $\tilde{N}^{(r3)} \simeq \mathcal{O}(1)$. In any of the $2n$ terms one gets after commuting the prefactor through the oscillators there is precisely one oscillator with reversed worldsheet momentum. This oscillator can now contract with any oscillator in the single string state or two-string state depending on whether the reversed oscillator belongs to the two-string or single string state. For each such contraction there is a corresponding one in which the sign of the worldsheet momentum of the two oscillators involved in the contraction is reversed[17]. The combination of these two contractions we represent by the vertex $(r, m) \overset{}{\text{———}\!\!\times\!\!\text{———}} (3, l)$ in (3.9), where $\times$ signifies the action of the prefactor on the oscillators $\alpha_{m(r)}$ and $\alpha_{l(3)}$. These two terms combine to yield an expression of $\mathcal{O}(1/\mu^2)$ due to the leading cancellation of the energy difference $\left(\frac{\omega_{m(r)}}{\mu p^+_{(r)}} - \frac{\omega_{l(3)}}{\mu}\right)$ of these two oscillators in the large μ limit. Therefore, this class of diagrams yields an expression given by the product of n Neumann matrices of the type $\tilde{N}^{(r3)}$ for $r = 1$ or 2 times the energy difference between one oscillator in the single string state and one oscillator in the two-string state.

[17] This appears from the term one gets after commuting the prefactor through the other oscillator.

We now consider the case with one self-contraction on the single string state. As mentioned above, and due to the impurity conservation condition, this self-contraction is always accompanied by a self-contraction on the two-string state. Therefore we have a contribution of the form $\tilde{N}^{(33)} \cdot \tilde{N}^{(rs)}$, where r, s is 1 or 2, which is already of order $\mathcal{O}(1/\mu^2)$. There are now two possibilities to be considered. Either any of the oscillators involved in the self-contraction have their worldsheet momentum reversed due to action of the prefactor or they don't. If they do not, then there is a contraction connecting the single string state with the two-string state involving the oscillator with the worldsheet momentum reversed. Just as in the previous case of no self-contractions, such diagram always comes accompanied with another one in which the sign of the worldsheet momentum is reversed on both oscillators involved in the contraction, yielding the vertex $(r,m) \overset{}{-\!\!\!-\!\!\!\times\!\!\!-\!\!\!-} (3,l)$ for $r=1$ or 2. Therefore, in this case, the diagram is proportional to $\left(\frac{\omega_{m(r)}}{\mu p^+_{(r)}} - \frac{\omega_{l(3)}}{\mu}\right) \cdot \tilde{N}^{(33)} \cdot \tilde{N}^{(rs)} \simeq \mathcal{O}(1/\mu^4)$, so it does not contribute to the leading order result. Therefore, in the case of one self-contraction the only possibility left is the case in which the self-contractions involve one oscillator which has the worldsheet momentum reversed due to the prefactor, so that only diagrams with the vertex $(3,m) -\!\!\!-\!\!\!\times\!\!\!-\!\!\!- (3,l)$ or $(r,m) -\!\!\!-\!\!\!\times\!\!\!-\!\!\!- (s,l)$ for $r, s = 1$ or 2 contribute to the leading order result.

From now on, let us focus on a particular Feynman diagram and show agreement between the string field theory and gauge theory computation. The string states with n impurities that we need to consider are given by[18]

$$
\begin{aligned}
|(d_i, n_i)\rangle &= i^n \delta_{\sum_i n_i, 0} \prod_i \alpha^{d_i \,\dagger}_{n_i} |\text{vac}\rangle, \\
|(e_i, p_i); \mathcal{I}_1, \mathcal{I}_2; y\rangle\rangle &= i^n \delta_{\sum_{j\in\mathcal{I}_1} p_j, 0} \delta_{\sum_{k\in\mathcal{I}_2} p_k, 0} \prod_{j\in\mathcal{I}_1} \alpha^{e_j \,\dagger}_{p_j} |\text{vac}, y\rangle \otimes \prod_{k\in\mathcal{I}_2} \alpha^{e_k \,\dagger}_{p_k} |\text{vac}, 1-y\rangle,
\end{aligned}
\tag{4.1}
$$

where the δ-functions impose the familiar level matching condition. The corresponding

[18] The arbitrary phase of the state is determined by comparison with gauge theory.

level-matched gauge theory operators are given by[19]:

$$\mathcal{O}^J_{(d_i,n_i)} = \frac{1}{\sqrt{JN^{J+n}}} \sum_{0\le l_1,\cdots,l_n\le J} \mathrm{Tr}\left(Z\ldots Z\frac{\phi_{d_1}}{\sqrt{J}} Z\ldots Z\frac{\phi_{d_2}}{\sqrt{J}} Z\ldots Z\frac{\phi_{d_n}}{\sqrt{J}} Z\ldots Z\right) \prod_{i=1}^{n} t_i^{l_i}$$

$$+ \text{ terms involving } \bar{Z} \qquad\qquad \text{with} \quad \sum_{i=1}^{n} n_i = 0,$$

$$\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2} =: \mathcal{O}^{y\cdot J}_{(e_j,p_j)_{j\in\mathcal{I}_1}} \cdot \mathcal{O}^{(1-y)\cdot J}_{(e_k,p_k)_{k\in\mathcal{I}_2}} : \qquad \text{with} \quad \sum_{j\in\mathcal{I}_1} p_j = \sum_{k\in\mathcal{I}_2} p_k = 0, \tag{4.2}$$

The labels $d_i, e_i \in \{1,2,3,4\}$ denote the direction along $\mathbf{R}^4$ of a string oscillator and the corresponding gauge theory impurity, and $n_i, p_i \in \mathbf{Z}$ are their worldsheet momenta where $t_i = \exp(2\pi i n_i/J)$, and $s_j = \exp(2\pi i p_j/J_1)$ for $j \in \mathcal{I}_1$ and $s_k = \exp(2\pi i p_k/J_2)$ for $k \in \mathcal{I}_2$. Also we explicitly assign a factor of $1/\sqrt{J}$ to each impurity which aids in keeping track of factors of J during the computation. $\mathcal{I}_1$ and $\mathcal{I}_2$ is a partition of the index set $\{1,\cdots,n\}$, which describes a particular way of distributing the n impurities among string/trace 1 and 2 respectively.

Let us explain the gauge theory computation of the two-point function of single-trace and double-trace BMN operators defined above and exhibit analogies with the string theory computation. At one loop order, that is to $\mathcal{O}(\lambda')$, we can have at most a quartic interaction[20] vertex, coupling four fields, with two of them contracted with the *in*-operator and the other two with the *out*-operator. There are three kinds of interaction vertices depending on how far the two fields in the same operator are separated:

• The nearest neighbor interaction[21] vertex where two fields on each operator coupled by the interaction sit next to each other, involves one impurity in the *in*-operator, and the same impurity and Z in the *out*-operator. This interaction can occur at $\mathcal{O}(J)$ sites along the smaller trace operator and we have to sum over the position of the interaction in the trace.

• The semi-nearest neighbor interaction vertex has two fields on one side sitting next to each other but the two fields on the other side are separated by $\mathcal{O}(J)$ sites. The vertex

[19] Here are using a simplified notation for the operators, their precise description is given in Appendix G and H.

[20] As shown in [45][14][15] the other possible interactions cancel among themselves due to supersymmetry.

[21] This terminology was first introduced in [14].

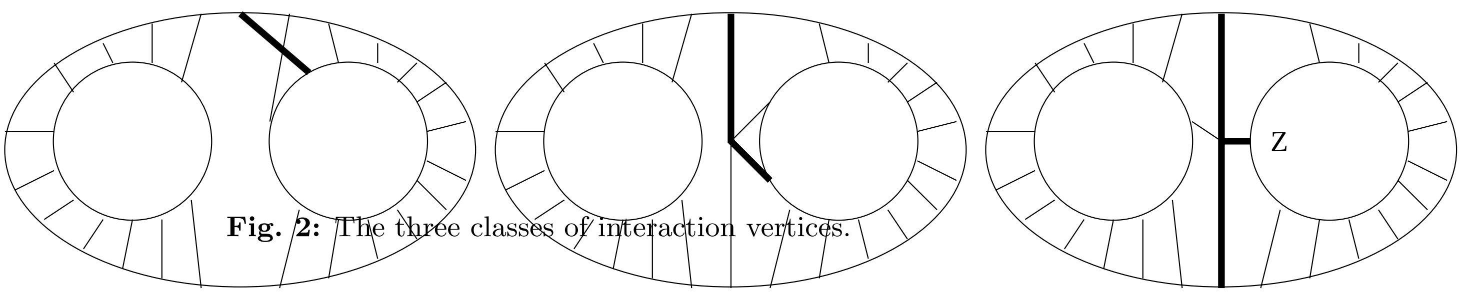

Fig. 2: The three classes of interaction vertices.

can be inserted only at a particular place along the trace and so we do not sum over the position of the vertex.

• The non-nearest neighbor interaction vertex has the two fields on each side of the interaction point separated by $\mathcal{O}(J)$ sites. In this vertex, two impurities or $\bar{Z}$ are involved in the two operators and this is possible only when we have two identical impurities in each operator. This interaction can also occur at a specific location in the trace, so we do not sum over the position of the vertex.

The contribution of each interaction vertex is given as:

$$\begin{aligned} I^{\text{nearest}}_{n_i,p_i}(l_i) = \frac{1}{\sqrt{JJ_1}}\frac{g^2N}{8\pi^2}(1-t_i)(1-\bar{s}_i)(t_i\bar{s}_i)^{l_i} \qquad &\text{for } i\in\mathcal{I}_1, \\ \text{or} \qquad\qquad\qquad\qquad & \qquad\qquad\qquad\qquad (4.3) \\ \frac{1}{\sqrt{JJ_2}}\frac{g^2N}{8\pi^2}(1-t_i)(1-\bar{s}_i)t_i^{J_1}(t_i\bar{s}_i)^{l_i} \qquad &\text{for } i\in\mathcal{I}_2, \end{aligned}$$

$$\begin{aligned} I^{\text{semi-nearest}}_{n_i,p_i} &= -\frac{1}{\sqrt{JJ_1}}\frac{g^2N}{8\pi^2}[(1-t_i)+(1-\bar{s}_i)](1-t_i^{J_1}) \qquad \text{for } i\in\mathcal{I}_1, \\ &\text{or} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.4) \\ &\frac{1}{\sqrt{JJ_2}}\frac{g^2N}{8\pi^2}[(1-t_i)+(1-\bar{s}_i)](1-t_i^{-J_2}) \qquad \text{for } i\in\mathcal{I}_2, \end{aligned}$$

$$\begin{aligned} I^{\text{non-nearest}}_{n_i,n_j,p_i,p_j} &= -\frac{1}{\sqrt{JJJ_1J_1}}\frac{g^2N}{8\pi^2}(1-t_i^{J_1})(1-t_j^{J_1}) \qquad \text{for } i,j\in\mathcal{I}_1, \\ &\text{or} \\ &-\frac{1}{\sqrt{JJJ_2J_2}}\frac{g^2N}{8\pi^2}(1-t_i^{J_1})(1-t_j^{J_1}) \qquad \text{for } i,j\in\mathcal{I}_2, \qquad (4.5) \\ &\text{or} \\ &\frac{1}{\sqrt{JJJ_1J_2}}\frac{g^2N}{8\pi^2}(1-t_i^{J_1})(1-t_j^{J_1}) \qquad \text{for } i\in\mathcal{I}_1, j\in\mathcal{I}_2, \end{aligned}$$

where l_i in $I^{\text{nearest}}_{n_i,p_i}(l_i)$ denotes the position of the nearest neighbor interaction vertex to be summed over. Here each factor of $1/\sqrt{J}$ or $1/\sqrt{J_r}$ $(r=1,2)$ comes from each impurity participating in the interaction. The rest of impurities in the *in*-operator are freely contracted with the remaining impurities in the *out*-operator and each free contraction contributes

$$\frac{1}{\sqrt{JJ_1}}(t_i\bar{s}_i)^{l_i} \;\; \text{for} \;\; i\in\mathcal{I}_1 \qquad \text{or} \qquad \frac{1}{\sqrt{JJ_2}}t_i^{J_1}(t_i\bar{s}_i)^{l_1} \;\; \text{for} \;\; i\in\mathcal{I}_2. \qquad (4.6)$$

Now we have to multiply all the the different contributions, coming from the interaction vertex and the free contractions and sum over all possible positions of the impurities. However, the whole summation is simply factorized in the large J limit into sums over each contribution since each contribution is independent of the positions of the rest of impurities:

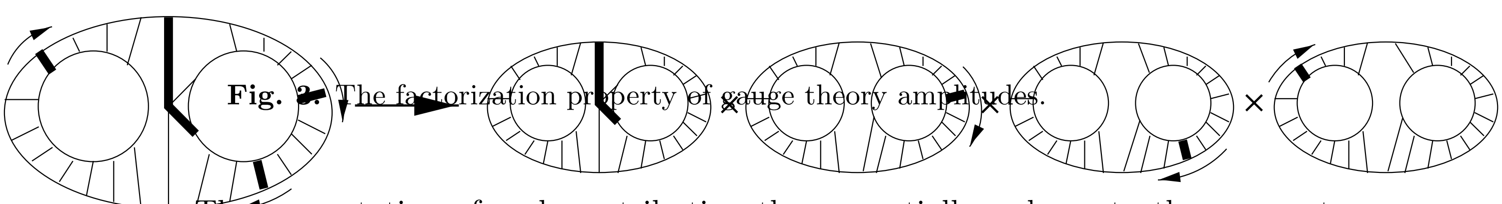

Fig. 3: The factorization property of gauge theory amplitudes.

The computation of each contribution then essentially reduces to the one or two-impurity cases. This factorization property, which as we have seen earlier has an analog in the string field theory computation, will turn out to be useful in comparing the gauge theory and string theory expressions for Feynman diagrams. As we will see, the effect of the prefactor interaction (r,m) ——×—— (s,l) in string field theory is essentially captured by the interaction vertex in gauge theory while the sum over free contractions in gauge theory capture the Neumann matrices.

Now, let us start to compute the string field theory amplitudes and compare them with the gauge theory results. As discussed earlier, there are two cases to be considered.

1) Case 1 : Diagrams without self-contraction

First, let us consider a particular way of contracting the oscillators without self-contractions. In this case, without loss of generality, we can assume that $d_i = e_i$ and take the d_i-th oscillator to contract with e_i-th oscillator for all $i \in \{1, \cdots, n\}$. More specifically, the j-th oscillator in string 3 contracts with the j-th oscillator in string 1 for $j \in \mathcal{I}_1$ and the k-th oscillator in string 3 contracts with the k-th oscillator in string 2 for $k \in \mathcal{I}_2$. On the string field theory side, using the Feynman rules in (3.9) we can compute the matrix elements between these states as in the previous section:

$$\begin{aligned}
&\frac{1}{\mu}\langle (d_i, n_i)|H_3|(d_i, p_i); \mathcal{I}_1, \mathcal{I}_2; y\rangle\rangle \\
&= -(-1)^n \frac{y(1-y)}{2}\left\{\sum_{l\in\mathcal{I}_1}\left[\left(\frac{\omega_{p_l(1)}}{\mu y} - \frac{\omega_{n_l(3)}}{\mu}\right)\tilde{N}^{(13)}_{p_l,-n_l}\prod_{j\in\mathcal{I}_1-\{l\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}\right]\right. \\
&\qquad\left. + \sum_{l\in\mathcal{I}_2}\left[\left(\frac{\omega_{p_l(2)}}{\mu(1-y)} - \frac{\omega_{n_l(3)}}{\mu}\right)\tilde{N}^{(23)}_{p_l,-n_l}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{l\}}\tilde{N}^{(23)}_{p_k,n_k}\right]\right\}.
\end{aligned} \tag{4.7}$$

Now let us explain how to match each term above with specific Feynman diagrams in gauge theory.

- $l \in \mathcal{I}_1$

For each $l \in \mathcal{I}_1$, the particular term

$$(-1)^n \frac{y(1-y)}{2}\left(\frac{\omega_{n_l(3)}}{\mu} - \frac{\omega_{p_l(1)}}{\mu y}\right)\tilde{N}^{(13)}_{p_l,-n_l}\prod_{j\in\mathcal{I}_1-\{l\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}, \tag{4.8}$$

arises when the l-th oscillator in string 1 and string 3 go through the prefactor and contract while the rest of the oscillators get contracted among themselves. The pair of l-th oscillators produce

$$\frac{1}{2}\left(\frac{\omega_{n_l(3)}}{\mu} - \frac{\omega_{p_l(1)}}{\mu y}\right)\tilde{N}^{(13)}_{p_l,-n_l} \simeq \frac{1}{4\mu^2}\left(n_l - \frac{p_l}{y}\right)^2\tilde{N}^{(13)}_{p_l,n_l}, \tag{4.9}$$

where we have used the large μ relation

$$\tilde{N}^{(r3)}_{p,-n} \simeq \frac{n - \frac{p}{y}}{n + \frac{p}{y}}\tilde{N}^{(r3)}_{p,n} \qquad r = 1 \text{ or } 2. \tag{4.10}$$

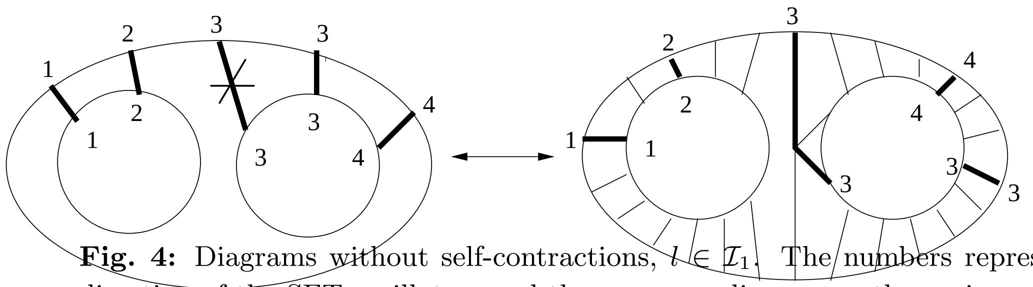

Fig. 4: Diagrams without self-contractions, $l \in \mathcal{I}_1$. The numbers represent the direction of the SFT oscillators and the corresponding gauge theory impurities.

The other pairs of oscillators bring down one Neumann coefficient $\tilde{N}^{(13)}_{p_j,n_j}$ or $\tilde{N}^{(23)}_{p_k,n_k}$. Therefore the contribution to the Hamiltonian matrix element due to this diagram is[22]:

$$\begin{aligned}
&\frac{1}{\mu}\langle (d_i,n_i)|H_3|(d_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle\Big|_l \\
&\simeq (-1)^n \frac{1}{4\mu^2}\sqrt{\frac{y(1-y)}{J}}\left(n_l - \frac{p_l}{y}\right)^2 \tilde{N}^{(13)}_{p_l,n_l} \prod_{j\in\mathcal{I}_1-\{l\}} \tilde{N}^{(13)}_{p_j,n_j} \prod_{k\in\mathcal{I}_2} \tilde{N}^{(23)}_{p_k,n_k}.
\end{aligned} \tag{4.11}$$

We claim that this particular term corresponds to the interaction Feynman diagrams where two ϕ_l's are involved in the interaction vertex and the rest of the impurities are freely contracted. The contributions come from two classes of diagrams. The nearest neighbor

[22] After going to the unit norm basis.

diagrams give

$$\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l^{\text{nearest}} = \sqrt{\frac{y(1-y)}{J}} \times \left[\frac{g^2N}{8\pi^2}(1-t_l)(1-\bar{s}_l)\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_l\bar{s}_l)^a\right]$$
$$\times \prod_{j\in\mathcal{I}_1-\{l\}}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a \prod_{k\in\mathcal{I}_2}\frac{1}{\sqrt{JJ_2}}\sum_{a=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^a, \tag{4.12}$$

whereas the semi-nearest neighbor diagrams contribute

$$\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l^{\text{semi-nearest}} = \sqrt{\frac{y(1-y)}{J}} \times \left[-\frac{1}{\sqrt{JJ_1}}\frac{g^2N}{8\pi^2}[(1-t_l)+(1-\bar{s}_l)](1-t_l^{J_1})\right]$$
$$\times \prod_{j\in\mathcal{I}_1-\{l\}}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a \prod_{k\in\mathcal{I}_2}\frac{1}{\sqrt{JJ_2}}\sum_{a=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^a, \tag{4.13}$$

where the phases are defined as $t_i = \exp(2\pi i n_i/J)$, and $s_j = \exp(2\pi i p_j/J_1)$ for $j \in \mathcal{I}_1$ and $s_k = \exp(2\pi i p_k/J_2)$ for $k \in \mathcal{I}_2$. The subscript l means that only Feynman diagrams with ϕ_l's involved in the interaction vertex are included. The first factor in (4.12) and (4.13) comes from the interaction vertices involving ϕ_l and the rest of the expression comes from free contraction of the other impurities. We can compute each factor and express it in terms of purely string field theory quantities and show that the interaction essentially captures the energy difference factor in the string theory computation while the free contractions yield the Neumann matrices. For $j \in \mathcal{I}_1$ and $k \in \mathcal{I}_2$, the free contraction contribution is:

$$\begin{aligned}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a &\simeq (-1)^{n_j+p_j+1}e^{i\pi n_j y}\tilde{N}^{(13)}_{p_j,n_j},\\ \frac{1}{\sqrt{JJ_2}}\sum_{a=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^a &\simeq (-1)^{n_k+1}e^{i\pi n_k y}\tilde{N}^{(23)}_{p_k,n_k},\end{aligned} \tag{4.14}$$

while the interaction vertex contribution is:

$$\begin{aligned}\frac{1}{\sqrt{JJ_1}}\frac{g^2N}{8\pi^2}(1-t_l)(1-\bar{s}_l)\sum_{a=0}^{J_1-1}(t_l\bar{s}_l)^a &\simeq (-1)^{n_l+p_l+1}e^{i\pi n_l y}\times\frac{\lambda'}{2}\left(\frac{n_l p_l}{y}\right)\tilde{N}^{(13)}_{p_l,n_l},\\ -\frac{1}{\sqrt{JJ_1}}\frac{g^2N}{8\pi^2}[(1-t_l)+(1-\bar{s}_l)](1-t_l^{J_1}) &\simeq (-1)^{n_l+p_l+1}e^{i\pi n_l y}\times\frac{\lambda'}{2}\left(n_l-\frac{p_l}{y}\right)^2\tilde{N}^{(13)}_{p_l,n_l}.\end{aligned} \tag{4.15}$$

Altogether, we obtain

$$\begin{aligned}
&\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l \\
&\simeq (-1)^n \frac{\lambda'}{2}\sqrt{\frac{y(1-y)}{J}}\left[\left(n_l-\frac{p_l}{y}\right)^2+n_l\frac{p_l}{y}\right]\tilde{N}^{(13)}_{p_l,n_l}\prod_{j\in\mathcal{I}_1-\{l\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}.
\end{aligned} \tag{4.16}$$

Notice that all the phase factors except $(-1)^n$ disappear upon imposing the level-matching conditions. In order to compare with the string theory result, we must evaluate these expressions in the string field theory basis (2.4). In order to compute,

$$\tilde{\Gamma}^{(1)}\Big|_l = \Gamma^{(1)}\Big|_l - \frac{1}{2}\{G^{(1)},\Gamma^{(0)}\Big|_l\}, \tag{4.17}$$

we also need to compute $G^{(1)}$ and $\Gamma^{(0)}\Big|_l$. They are given by[23]:

$$\begin{aligned}
G^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y} &= \sqrt{\frac{y(1-y)}{J}}\prod_{j\in\mathcal{I}_1}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a\prod_{k\in\mathcal{I}_2}\frac{1}{\sqrt{JJ_2}}\sum_{a=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^a \\
&\simeq (-1)^n\sqrt{\frac{y(1-y)}{J}}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}, \\
\Gamma^{(0)}_{\{n_i\},\{m_i\}}\Big|_l &= \frac{\lambda'}{2}n_l^2\prod_i\delta_{n_i,m_i}, \\
\Gamma^{(0)}_{\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y,\{q_i;\mathcal{I}_1,\mathcal{I}_2\}z}\Big|_l &= \frac{\lambda'}{2}\left(\frac{p_l}{y}\right)^2\delta_{y,z}\prod_i\delta_{p_i,q_i}.
\end{aligned} \tag{4.18}$$

Hence,

$$\tilde{\Gamma}^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l \simeq (-1)^n\frac{\lambda'}{4}\sqrt{\frac{y(1-y)}{J}}\left(n_l-\frac{p_l}{y}\right)^2\tilde{N}^{(13)}_{p_l,n_l}\prod_{j\in\mathcal{I}_1-\{l\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}. \tag{4.19}$$

which precisely reproduces the string field theory result (4.11).

- $l\in\mathcal{I}_2$

[23] Here we use the large μ relation (4.14) to rewrite $G^{(1)}$ in terms of string field theory quantities.

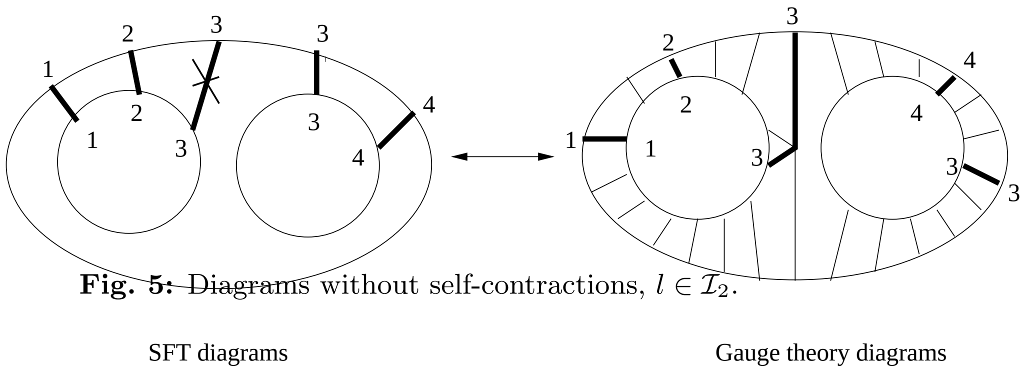

Fig. 5: Diagrams without self-contractions, $l \in \mathcal{I}_2$.

Now we consider the other type of contraction in the string field theory computation, where the prefactor acts on the l-th oscillator in string 2 and string 3. The expression for this diagram is:

$$(-1)^n \frac{y(1-y)}{2} \left(\frac{\omega_{n_l(3)}}{\mu} - \frac{\omega_{p_l(2)}}{\mu(1-y)} \right) \tilde{N}^{(23)}_{p_l,-n_l} \prod_{j \in \mathcal{I}_1} \tilde{N}^{(13)}_{p_j,n_j} \prod_{k \in \mathcal{I}_2 - \{l\}} \tilde{N}^{(23)}_{p_k,n_k} \,. \tag{4.20}$$

As before, it is convenient to express the contribution from the prefactor as:

$$\frac{1}{2} \left(\frac{\omega_{n_l(3)}}{\mu} - \frac{\omega_{p_l(2)}}{\mu(1-y)} \right) \tilde{N}^{(23)}_{p_l,-n_l} \simeq \frac{1}{4\mu^2} \left(n_l - \frac{p_l}{1-y} \right)^2 \tilde{N}^{(23)}_{p_l,n_l}, \tag{4.21}$$

where we have used the large μ relation (4.10). Therefore, the contribution of this diagram to the Hamiltonian matrix element of unit normalized states is:

$$
\begin{aligned}
&\frac{1}{\mu}\langle(d_i,n_i)|H_3|(d_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle\Big|_l \\
&\simeq(-1)^n\frac{1}{4\mu^2}\sqrt{\frac{y(1-y)}{J}}\left(n_l-\frac{p_l}{1-y}\right)^2\tilde{N}^{(23)}_{p_l,n_l}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{l\}}\tilde{N}^{(23)}_{p_k,n_k}.
\end{aligned}
\tag{4.22}
$$

The corresponding gauge theory diagrams are again classified into two classes. The nearest neighbor diagrams yields

$$
\begin{aligned}
\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l^{\text{nearest}}&=\sqrt{\frac{y(1-y)}{J}}\times\left[\frac{g^2N}{8\pi^2}(1-t_l)(1-\bar{s}_l)\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_l^{J_1}(t_l\bar{s}_l)^b\right]\\
&\times\prod_{j\in\mathcal{I}_1}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a\prod_{k\in\mathcal{I}_2-\{l\}}\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^b,
\end{aligned}
\tag{4.23}
$$

whereas the semi-nearest neighbor diagrams contribute

$$
\begin{aligned}
\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l^{\text{semi-nearest}}&=\sqrt{\frac{y(1-y)}{J}}\times\left[\frac{1}{\sqrt{JJ_2}}\frac{g^2N}{8\pi^2}[(1-t_l)+(1-\bar{s}_l)](1-t_l^{J_1})\right]\\
&\times\prod_{j\in\mathcal{I}_1}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a\prod_{k\in\mathcal{I}_2-\{l\}}\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^b.
\end{aligned}
\tag{4.24}
$$

We can also express the various contributions in terms of string field theory quantities. The interaction vertex contribution is given by

$$
\begin{aligned}
\frac{g^2N}{8\pi^2}(1-t_l)(1-\bar{s}_l)\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_l^{J_1}(t_l\bar{s}_l)^b&\simeq(-1)^{n_l+1}e^{i\pi n_l y}\times\frac{\lambda'}{2}\left(\frac{n_lp_l}{1-y}\right)\tilde{N}^{(23)}_{p_l,n_l},\\
\frac{1}{\sqrt{JJ_2}}\frac{g^2N}{8\pi^2}[(1-t_l)+(1-\bar{s}_l)](1-t_l^{J_1})&\simeq(-1)^{n_l+1}e^{i\pi n_l y}\times\frac{\lambda'}{2}\left(n_l-\frac{p_l}{1-y}\right)^2\tilde{N}^{(23)}_{p_l,n_l},
\end{aligned}
\tag{4.25}
$$

whereas the free contraction (4.14) yields the product of Neumann matrix after imposing the level matching constraint.

In order to compute the matrix of anomalous dimensions in the string field theory basis we need also $G^{(1)}$ and $\Gamma^{(0)}\Big|_l$. It is easy to show that these quantities are the same as in (4.18) except for the last formula which can be correctly obtained by replacing $y\to 1-y$. Therefore, using (4.17), we obtain:

$$
\tilde{\Gamma}^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|_l\simeq(-1)^n\frac{\lambda'}{4}\sqrt{\frac{y(1-y)}{J}}\left(n_l-\frac{p_l}{1-y}\right)^2\tilde{N}^{(23)}_{p_l,n_l}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{l\}}\tilde{N}^{(23)}_{p_k,n_k},
\tag{4.26}
$$

and again we find agreement with the string theory result (4.22).

2) Terms with self-contractions

As explained in the beginning of this section, to leading order in the $1/\mu$ expansion we can have at most one self-contraction in string 3 and the prefactor has to go through any of the oscillators involved in the self-contraction.

Without loss of generality, we can assume that $d_1 = d_2$, $e_1 = e_2$, $d_i = e_i$ for $i \in \{3, \cdots, n\}$ and we will consider contractions between $d_1 - d_2$, $e_1 - e_2$, and $d_i - e_i$ for $i \in \{3, \cdots, n\}$. There are three cases depending on how the 1st and the 2nd impurities are distributed on the two-string state and the double-trace operator: $1, 2 \in \mathcal{I}_1$, $1, 2 \in \mathcal{I}_2$ and $1 \in \mathcal{I}_1$, $2 \in \mathcal{I}_2$.

- $1, 2 \in \mathcal{I}_1$

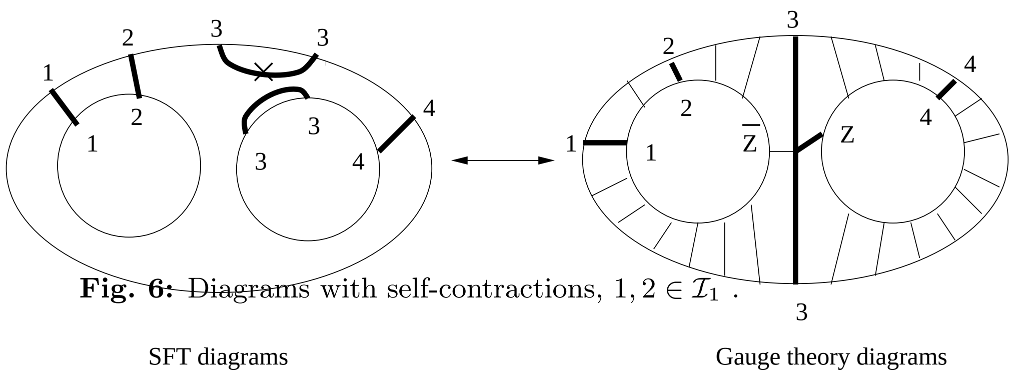

Fig. 6: Diagrams with self-contractions, $1, 2 \in \mathcal{I}_1$.

The string theory computation of this particular Feynman diagram is:

$$\begin{aligned}
&\frac{1}{\mu}\langle (d_i,n_i)|H_3|(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle \\
&= -(-1)^n\frac{y(1-y)}{2}\left[\left(\frac{\omega_{p_1(1)}+\omega_{p_2(1)}}{\mu y}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,-p_2}-\left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(11)}_{p_1,p_2}\right] \\
&\quad\times\prod_{j\in\mathcal{I}_1-\{1,2\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}.
\end{aligned} \tag{4.27}$$

The first line in (4.27) is due to the self-contractions while the rest is due to the contraction between oscillators in the single string state with the two-string state.

The self-contraction contribution is to the leading order in $1/\mu$:

$$\begin{aligned}
&-\frac{1}{2}\left[\left(\frac{\omega_{p_1(1)}+\omega_{p_2(1)}}{\mu y}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,-p_2}-\left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(11)}_{p_1,p_2}\right] \\
&\simeq -2\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,p_2}.
\end{aligned} \tag{4.28}$$

Therefore, the matrix element of unit normalized states is given by

$$\begin{aligned}
&\frac{1}{\mu}\langle (d_i,n_i)|H_3|(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle\Big|_{1-2;1-2} \\
&\simeq -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1-\{1,2\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}.
\end{aligned} \tag{4.29}$$

We now show that the corresponding gauge theory diagrams are those with an interaction vertex involving ϕ_{d_1},ϕ_{d_2} or $\bar{Z}$ in $\mathcal{O}^J_{(d_i,n_i)}$ and ϕ_{e_1},ϕ_{e_2} or $\bar{Z}$ in $\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2}$. In this case, only non-nearest interaction diagrams contribute and the result is:

$$\begin{aligned}
\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\mathrm{non-nearest}}_{1-2;1-2} &= \sqrt{\frac{y(1-y)}{J}}\times\left[-\frac{1}{\sqrt{JJJ_1J_1}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1})\right] \\
&\quad\times\prod_{j\in\mathcal{I}_1-\{1,2\}}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a\prod_{k\in\mathcal{I}_2}\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^b.
\end{aligned} \tag{4.30}$$

The interaction contribution reduces in the BMN limit to

$$\begin{aligned}
-\frac{1}{\sqrt{JJJ_1J_1}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1}) &= e^{\pi i(n_1+n_2)y}\lambda'\frac{\sin(\pi n_1 y)\sin(\pi n_2 y)}{2\pi^2 y} \\
&\simeq -2(-1)^{n_1+n_2+p_1+p_2}e^{\pi i(n_1+n_2)y}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,p_2},
\end{aligned} \tag{4.31}$$

while the rest can be rewritten in string field theory language using (4.14). Again the various phase factors disappear after imposing the level matching condition on each trace.

In order to compare with string field theory we must go to the string field theory basis. However, the particular class of Feynman diagrams we are considering, which are those with an interaction vertex involving ϕ_{d_1}, ϕ_{d_2} or $\bar{Z}$ in $\mathcal{O}^J_{(d_i,n_i)}$ and ϕ_{e_1}, ϕ_{e_2} or $\bar{Z}$ in $\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2}$ do not contribute to $\Gamma^{(0)}\Big|_l$. Therefore, in this case (4.17) yields:

$$\tilde{\Gamma}^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Bigg|^{\mathrm{non-nearest}}_{1-2;1-2} \simeq -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(11)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1-\{1,2\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2}\tilde{N}^{(23)}_{p_k,n_k}, \tag{4.32}$$

which agrees with the SFT result (4.27).

• $1, 2 \in \mathcal{I}_2$

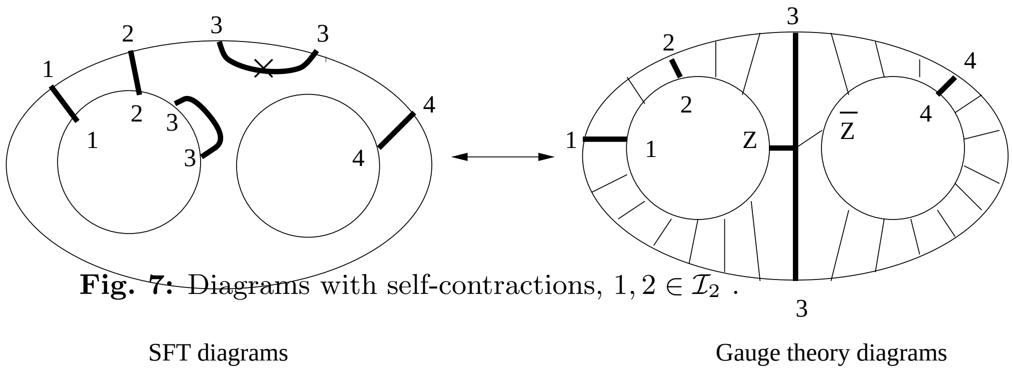

Fig. 7: Diagrams with self-contractions, $1, 2 \in \mathcal{I}_2$.

The string theory computation of this particular Feynman diagram is similar to the previous one:

$$\begin{aligned}
&\frac{1}{\mu}\langle (d_i, n_i)|H_3|(e_i, p_i); \mathcal{I}_1, \mathcal{I}_2; y\rangle\rangle \\
&= -(-1)^n \frac{y(1-y)}{2}\left[\left(\frac{\omega_{p_1(2)}+\omega_{p_2(2)}}{\mu(1-y)}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,-p_2} - \left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(22)}_{p_1,p_2}\right] \\
&\quad \times \prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{1,2\}}\tilde{N}^{(23)}_{p_k,n_k}.
\end{aligned} \tag{4.33}$$

The self-contraction contribution is to the leading order in $1/\mu$:

$$\begin{aligned}
&-\frac{1}{2}\left[\left(\frac{\omega_{p_1(2)}+\omega_{p_2(2)}}{\mu(1-y)}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,-p_2}-\left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(22)}_{p_1,p_2}\right] \\
&\simeq -2\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,p_2}.
\end{aligned} \tag{4.34}$$

Therefore, the matrix element of unit normalized states is given by

$$\begin{aligned}
&\frac{1}{\mu}\langle(d_i,n_i)|H_3|(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle\Big|_{1-2;1-2} \\
&\simeq -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{1,2\}}\tilde{N}^{(23)}_{p_k,n_k}.
\end{aligned} \tag{4.35}$$

We now show that the corresponding gauge theory diagrams are those with an interaction vertex involving ϕ_{d_1},ϕ_{d_2} or $\bar{Z}$ in $\mathcal{O}^J_{(d_i,n_i)}$ and ϕ_{e_1},ϕ_{e_2} or $\bar{Z}$ in $\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2}$. In this case, only non-nearest interaction diagrams contribute and the result is:

$$\begin{aligned}
\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\text{non-nearest}}_{1-2;1-2} &= \sqrt{\frac{y(1-y)}{J}}\times\left[-\frac{1}{\sqrt{JJJ_2J_2}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1})\right] \\
&\times\prod_{j\in\mathcal{I}_1}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a\prod_{k\in\mathcal{I}_2-\{1,2\}}\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^b.
\end{aligned} \tag{4.36}$$

The interaction contribution reduces in the BMN limit to

$$\begin{aligned}
-\frac{1}{\sqrt{JJJ_2J_2}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1}) &= e^{\pi i(n_1+n_2)y}\lambda'\frac{\sin(\pi n_1y)\sin(\pi n_2y)}{2\pi^2(1-y)} \\
&\simeq -2(-1)^{n_1+n_2}e^{\pi i(n_1+n_2)y}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,p_2},
\end{aligned} \tag{4.37}$$

while the rest can be rewritten in string field theory language using (4.14). Again the various phase factors disappear after imposing the level matching condition on each trace.

In order to compare with string field theory we must go to the string field theory basis. However, the particular class of Feynman diagrams we are considering, which are those with an interaction vertex involving ϕ_{d_1},ϕ_{d_2} or $\bar{Z}$ in $\mathcal{O}^J_{(d_i,n_i)}$ and ϕ_{e_1},ϕ_{e_2} or $\bar{Z}$ in $\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2}$ do not contribute to $\Gamma^{(0)}\Big|_l$. Therefore, in this case (4.17) yields:

$$\tilde{\Gamma}^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\text{non-nearest}}_{1-2;1-2} \simeq -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(22)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{1,2\}}\tilde{N}^{(23)}_{p_k,n_k}, \tag{4.38}$$

which agrees with the SFT result (4.33).

- $1\in\mathcal{I}_1$, $2\in\mathcal{I}_2$

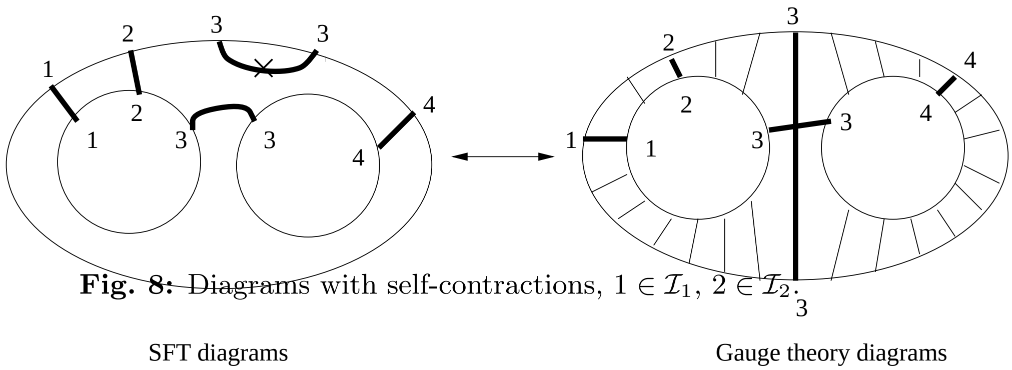

Fig. 8: Diagrams with self-contractions, $1 \in \mathcal{I}_1$, $2 \in \mathcal{I}_2$.

The string field theory computation of this particular contraction term is:

$$
\begin{aligned}
&\frac{1}{\mu}\langle (d_i, n_i)|H_3|(e_i, p_i); \mathcal{I}_1, \mathcal{I}_2; y\rangle\rangle \\
&= -(-1)^n \frac{y(1-y)}{2}\left[\left(\frac{\omega_{p_1(1)}}{\mu y} + \frac{\omega_{p_2(2)}}{\mu(1-y)}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,-p_2} - \left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(12)}_{p_1,p_2}\right] \\
&\quad \times \prod_{j\in\mathcal{I}_1-\{1\}} \tilde{N}^{(13)}_{p_j,n_j} \prod_{k\in\mathcal{I}_2-\{2\}} \tilde{N}^{(23)}_{p_k,n_k}.
\end{aligned}
\tag{4.39}
$$

The first factor which is the result of the self-contraction between the single and two-string state, is to the leading order in $1/\mu$:

$$
\begin{aligned}
&-\frac{1}{2}\left[\left(\frac{\omega_{p_1(1)}}{\mu y} + \frac{\omega_{p_2(2)}}{\mu(1-y)}\right)\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,-p_2} - \left(\frac{\omega_{n_1(3)}+\omega_{n_2(3)}}{\mu}\right)\tilde{N}^{(33)}_{n_1,-n_2}\tilde{N}^{(12)}_{p_1,p_2}\right] \\
&\simeq -2\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,p_2}.
\end{aligned}
\tag{4.40}
$$

Therefore, the contribution of this Feynman diagram to the matrix element of unit normalized states is given by:

$$
\begin{aligned}
&\frac{1}{\mu}\langle(d_i,n_i)|H_3|(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2;y\rangle\rangle\Big|_{1-2;1-2} \\
&\simeq -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1-\{1\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{2\}}\tilde{N}^{(23)}_{p_k,n_k}.
\end{aligned}
\tag{4.41}
$$

Now let us compute the corresponding gauge theory diagrams with an interaction vertex involving ϕ_{d_1},ϕ_{d_2} or $\bar{Z}$ in $\mathcal{O}^J_{(d_i,n_i)}$ and ϕ_{e_1},ϕ_{e_2} or $\bar{Z}$ in $\mathcal{T}^{J,y}_{(e_i,p_i);\mathcal{I}_1,\mathcal{I}_2}$. The result is:

$$
\begin{aligned}
\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\mathrm{semi-nearest}}_{1-2;1-2} &= \sqrt{\frac{y(1-y)}{J}}\times\left[\frac{1}{\sqrt{JJJ_1J_2}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1})\right] \\
&\times\prod_{j\in\mathcal{I}_1-\{1\}}\frac{1}{\sqrt{JJ_1}}\sum_{a=0}^{J_1-1}(t_j\bar{s}_j)^a\prod_{k\in\mathcal{I}_2-\{2\}}\frac{1}{\sqrt{JJ_2}}\sum_{b=0}^{J_2-1}t_k^{J_1}(t_k\bar{s}_k)^b.
\end{aligned}
\tag{4.42}
$$

The interaction part of the diagram reduces to

$$
\begin{aligned}
\frac{1}{\sqrt{JJJ_1J_2}}\frac{g^2N}{8\pi^2}(1-t_1^{J_1})(1-t_2^{J_1}) &= -e^{\pi i(n_1+n_2)y}\lambda'\frac{\sin(\pi n_1 y)\sin(\pi n_2 y)}{2\pi^2\sqrt{y(1-y)}} \\
&\simeq -2(-1)^{n_1+n_2+p_1}e^{\pi i(n_1+n_2)y}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,p_2},
\end{aligned}
\tag{4.43}
$$

while the rest of the diagram, the free contraction contribution, can be computed using (4.14) making the phase disappear after imposing the level matching condition on each trace.

Just as in the previous case, the Feynman diagrams we are considering do not contribute to $\Gamma^{(0)}\Big|_l$ so that their contribution to the matrix of anomalous dimensions in the string field theory basis is given by

$$
\tilde{\Gamma}^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}\Big|^{\mathrm{semi-nearest}}_{1-2;1-2} \simeq -2(-1)^n\sqrt{\frac{y(1-y)}{J}}\tilde{N}^{(33)}_{n_1,n_2}\tilde{N}^{(12)}_{p_1,p_2}\prod_{j\in\mathcal{I}_1-\{1\}}\tilde{N}^{(13)}_{p_j,n_j}\prod_{k\in\mathcal{I}_2-\{2\}}\tilde{N}^{(23)}_{p_k,n_k},
\tag{4.44}
$$

which agrees with the string theory result (4.41).

5. Conclusion

In this paper we have computed string interactions between string states with an arbitrary number of scalar impurities. Using the holographic map proposed in [7][8] and the basis of gauge theory states in [8] we have exactly reproduced all string amplitudes from gauge theory considerations. The calculations have been carried up to $\mathcal{O}(g_2^2)$ for the case of two identical impurities and to $\mathcal{O}(g_2)$ for arbitrary impurities. The precise agreement found gives strong support to the validity of the holographic map (1.2) and the basis of gauge theory states in [8]. The $\mathcal{O}(g_2^2)$ computation has been performed in string field theory by truncating [44] by hand the allowed intermediate states. With this truncation we get precise agreement with the gauge theory calculation. It is desirable to understand whether the truncation is necessary.

While considering arbitrary string states, we have found that there is a direct correspondence between the Feynman diagrams of gauge theory and the string field theory Feynman diagrams that contribute to a given amplitude. This diagrammatic correspondence is specially powerful when we consider general string states, in which new classes of Feynman diagrams appear as compared to the case with two different impurities. In particular we have shown which interaction vertex in gauge theory corresponds to which string field theory vertex arising from the action of the prefactor. Likewise the various Neumann matrices in string theory have been derived from purely field theoretic considerations as arising from various free contractions in gauge theory.

The diagrammatic equivalence between gauge theory one loop diagrams and string theory diagrams might be useful in deriving the duality, which is an important open problem. It would also be very desirable to represent the degrees of freedom of the BMN sector of $\mathcal{N} = 4$ SYM by a complete theory, without any truncation. Holography strongly suggests that there should be a quantum mechanical model which describes the BMN sector of $\mathcal{N} = 4$ SYM and at the same time captures all the physics of string theory in the plane wave geometry. An important step in this direction is the string bit model of Verlinde [10], but a suitable non-abelian generalization of it remains to be discovered.

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Appendix A. Asymptotic behavior of Neumann matrices

In this appendix we present the asymptotic large μ behavior of all Neumann matrices in the exponential basis. These can be obtained from $(m, n \neq 0)$

$$\tilde{N}^{(rs)}_{m,n} = \frac{1}{2}(\bar{N}^{(rs)}_{|m|,|n|} - e(mn)\bar{N}^{(rs)}_{-|m|,-|n|}), \qquad \tilde{N}^{(rs)}_{m,0} = \frac{1}{\sqrt{2}}\bar{N}^{(rs)}_{|m|,0}, \qquad \tilde{N}^{(rs)}_{0,0} = \bar{N}^{(rs)}_{0,0}, \quad \text{(A.1)}$$

where $e(m) = \mathrm{sign}(m)$ and the asymptotic behavior of Neumann matrices in the cos/sin basis in [42]:

$$\begin{aligned}
\tilde{N}^{(11)}_{m,n} &\simeq \frac{(-1)^{m+n}}{4\pi\mu y} \\
\tilde{N}^{(12)}_{m,n} &\simeq \frac{(-1)^{m+1}}{4\pi\mu\sqrt{y(1-y)}} \\
\tilde{N}^{(22)}_{m,n} &\simeq \frac{1}{4\pi\mu(1-y)} \\
\tilde{N}^{(13)}_{m,n} &\simeq \frac{(-1)^{m+n+1}\sin n\pi y}{\pi\sqrt{y}(n-m/y)} \\
\tilde{N}^{(23)}_{m,n} &\simeq \frac{(-1)^{n}\sin n\pi y}{\pi\sqrt{1-y}(n-m/(1-y))} \\
\tilde{N}^{(33)}_{m,n} &\simeq \frac{(-1)^{m+n+1}\sin m\pi y \sin n\pi y}{\pi\mu}.
\end{aligned} \qquad \text{(A.2)}$$

For the computation of the contact term, we also need $\tilde{F}^{\pm}$ in the exponential basis $(n \neq 0)$

$$\tilde{F}^{\pm}_{n(r)} = \frac{1}{\sqrt{2}}F^{\pm}_{|n|(r)}, \qquad \tilde{F}^{\pm}_{0(r)} = F^{\pm}_{0(r)}. \qquad \text{(A.3)}$$

and the scalar quantity k and fermionic Neumann matrices $\bar{Y}$. Using again the results in [42], we have

$$\begin{aligned}
\tilde{F}^{+}_{(1)n} &\simeq (-1)^{n+1}\sqrt{\mu y}(1-y) \\
\tilde{F}^{+}_{(2)n} &\simeq \sqrt{\mu(1-y)}\,y \\
\tilde{F}^{+}_{(3)n} &\simeq \frac{(-1)^{n+1} n y(1-y)\sin\pi n y}{\sqrt{\mu}},
\end{aligned} \qquad \text{(A.4)}$$

$$\begin{aligned}
\tilde{F}^{-}_{(1)n} &\simeq \frac{(-1)^{n+1} n(1-y)}{2\sqrt{\mu y}} \\
\tilde{F}^{-}_{(2)n} &\simeq \frac{ny}{2\sqrt{\mu(1-y)}} \\
\tilde{F}^{-}_{(3)n} &\simeq 2\sqrt{\mu}y(1-y)(-1)^{n+1}\sin\pi n y.
\end{aligned} \qquad \text{(A.5)}$$

$$1 - \mu y(1-y)k \simeq \frac{1}{4\pi\mu y(1-y)}, \tag{A.6}$$

$$\bar{Y}_0 \simeq \frac{1}{\sqrt{4\pi\mu y(1-y)}}, \qquad \bar{Y}_{n(1)} \simeq \sqrt{\frac{1-y}{4\pi\mu}}(-1)^{n+1}, \qquad \bar{Y}_{n(2)} \simeq \sqrt{\frac{y}{4\pi\mu}}. \tag{A.7}$$

Appendix B. Matrix elements

The definition of various matrices appearing in G and Γ on the gauge theory calculation are given as follows.

2-impurity matrix elements ($|m| \neq |n|, m \neq 0, n \neq 0, p \in Z, 0 < y < 1$)

$$\begin{aligned}
\bullet\quad & C_{n,py} = C_{py,n} = \frac{y^{3/2}\sqrt{1-y}}{\sqrt{J}\pi^2}\frac{\sin^2(\pi n y)}{(p-ny)^2} \\
& C_{n,y} = C_{y,n} = -\frac{1}{\sqrt{J}\pi^2}\frac{\sin^2(\pi n y)}{n^2} \\
\bullet\quad & M^1_{n,n} = \frac{1}{60} - \frac{1}{24\pi^2 n^2} + \frac{7}{16\pi^4 n^4} \\
& M^1_{n,-n} = \frac{1}{48\pi^2 n^2} + \frac{35}{128\pi^4 n^4} \\
& M^1_{n,m} = \frac{1}{12\pi^2(n-m)^2} - \frac{1}{8\pi^4(n-m)^4} + \frac{1}{4\pi^4 n^2 m^2} + \frac{1}{8\pi^4 nm(n-m)^2} \\
\bullet\quad & \Gamma^{(1)}_{n,py} = \Gamma^{(1)}_{py,n} = \lambda'\left(\frac{p^2}{y^2} - \frac{pn}{y} + n^2\right) C_{n,py} \\
& \Gamma^{(1)}_{n,y} = \Gamma^{(1)}_{y,n} = \lambda' n^2 C_{n,y} \\
& \Gamma^{(2)}_{n,m} = \lambda' nm M^1_{n,m} + \frac{1}{8\pi^2}\mathcal{D}^1_{n,m} \\
\bullet\quad & \mathcal{D}^1_{n,n} = \mathcal{D}^1_{n,-n} = \lambda'\left(\frac{2}{3} + \frac{5}{\pi^2 n^2}\right) \\
& \mathcal{D}^1_{n,m} = \lambda'\left(\frac{2}{3} + \frac{2}{\pi^2 n^2} + \frac{2}{\pi^2 m^2}\right) \\
\bullet\quad & B_{n,n} = \frac{1}{3} + \frac{5}{2\pi^2 n^2} \\
& B_{n,-n} = -\frac{15}{8\pi^2 n^2} \\
& B_{n,m} = \frac{3}{2\pi^2 mn} + \frac{1}{2\pi^2(m-n)^2}
\end{aligned} \tag{B.1}$$

n-impurity matrix elements

- $G^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y} = G^{(1)}_{\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y,\{n_i\}}$
  $= (-1)^{n+\sum_{k\in\mathcal{I}_2} n_k} \frac{\sqrt{y^{n_1+1}}\sqrt{(1-y)^{n_2+1}}}{\sqrt{J}} \prod_{j\in\mathcal{I}_1} \frac{\sin(\pi n_j y)}{\pi(p_j - n_j y)} \prod_{k\in\mathcal{I}_2} \frac{\sin(\pi n_k(1-y))}{\pi(p_k - n_k(1-y))},$
- $\Gamma^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y} = \Gamma^{(1)}_{\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y,\{n_i\}}$
  $= \frac{\lambda'}{2}\left[\sum_{j\in\mathcal{I}_1}\left(\left(n_j - \frac{p_j}{y}\right)^2 + n_j\frac{p_j}{y}\right) + \sum_{k\in\mathcal{I}_2}\left(\left(n_k - \frac{p_k}{1-y}\right)^2 + n_k\frac{p_k}{1-y}\right)\right]$
  $\times G^{(1)}_{\{n_i\},\{p_i;\mathcal{I}_1,\mathcal{I}_2\}y}.$

(B.2)

where $n_1 = |\mathcal{I}_1|, \ \ n_2 = |\mathcal{I}_2|$.

Appendix C. Calculation of supersymmetry charge matrix elements

In this appendix we shall show explicitly how to reduce the supersymmetry vertex in [41] to our simple formula (3.15) when we assume the external state to have two bosonic impurities and the intermediate state to have one bosonic and one fermionic impurity. See also [44].

The Hamiltonian and supersymmetry charge vertices in [41] are given by:

$$\begin{aligned}
|H_3\rangle &= c\Big((1+\mu\alpha k)(K^i_+ - K^i_-)(K^j_+ + K^j_-) - \mu\alpha\delta^{ij}\Big)v_{ij}(Y)E_aE_bE_{b0}|0\rangle, \\
|Q_{3\dot{a}}\rangle &= c(1+\mu\alpha k)^{1/2}(K^i_+ - K^i_-)s^i_{\dot{a}}(Y)E_aE_bE_{b0}|0\rangle, \\
|\bar{Q}_{3\dot{a}}\rangle &= c(1+\mu\alpha k)^{1/2}(K^i_+ + K^i_-)\tilde{s}^i_{\dot{a}}(Y)E_aE_bE_{b0}|0\rangle.
\end{aligned} \tag{C.1}$$

Various constituents of the prefactor, $K^i_\pm$, v^{ij} $s^i_{\dot{a}} = -i\sqrt{2}(\eta s^i_{1\dot{a}} + \bar{\eta}s^i_{2\dot{a}})$ and $\tilde{s}^i_{\dot{a}} = i\sqrt{2}(\bar{\eta}s^i_{1\dot{a}} + \eta s^i_{2\dot{a}})$ are given as

$$\begin{aligned}
K^i_+ &= \sum_{r=1}^{3}\sum_{m=-\infty}^{\infty} \tilde{F}^+_{m(r)}\alpha^{i\dagger}_{m(r)}, \\
K^i_- &= \sum_{r=1}^{3}\sum_{m=-\infty}^{\infty} \tilde{F}^-_{m(r)}\alpha^{i\dagger}_{m(r)},
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
v^{ij} &= \delta^{ij} + \frac{1}{4!\alpha^2}t^{ij}_{abcd}Y^aY^bY^cY^d + \frac{1}{8!\alpha^4}\delta^{ij}\epsilon_{abcdefgh}Y^a\cdots Y^h \\
&\quad + \frac{1}{2!\alpha}\gamma^{ij}_{ab}Y^aY^b + \frac{1}{2!6!\alpha^3}\gamma^{ij}_{ab}\epsilon^{ab}{}_{cdefgh}Y^c\cdots Y^h, \\
\frac{1}{\sqrt{2}}s^i_{1\dot{a}} &= \gamma^i_{a\dot{a}}Y^a + \frac{1}{3!5!\alpha^2}u^i_{abc\dot{a}}\epsilon^{abc}{}_{defgh}Y^d\cdots Y^h, \\
\frac{1}{\sqrt{2}}s^i_{2\dot{a}} &= -\frac{1}{3!\alpha}u^i_{abc\dot{a}}Y^aY^bY^c + \frac{1}{7!\alpha^3}\gamma^i_{a\dot{a}}\epsilon^a{}_{bcdefgh}Y^b\cdots Y^h,
\end{aligned} \tag{C.3}$$

where Y^a reads

$$Y^a = \sqrt{2}Y_0(\alpha_{(1)}\lambda^a_{(2)} - \alpha_{(2)}\lambda^a_{(1)}) + \sum_{r=1}^{3}\sum_{m=1}^{\infty} Y_{m(r)} b^{a\dagger}_{m(r)}, \tag{C.4}$$

with

$$\lambda^a_{(r)} = \sqrt{\frac{\alpha_{(r)}}{2}}\begin{pmatrix} b^{a\dagger}_{0(r)} \\ b^a_{0(r)} \end{pmatrix} \quad (r = 1, 2), \qquad\qquad \lambda^a_{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix} b^a_{0(3)} \\ b^{a\dagger}_{0(3)} \end{pmatrix}, \tag{C.5}$$

and

$$Y_0 = \bar{Y}_0\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} + \frac{1}{\bar{Y}_0}\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}, \tag{C.6}$$

$$Y_{n(1)} = \bar{Y}_{n(1)}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \qquad Y_{n(2)} = \bar{Y}_{n(2)}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}. \tag{C.7}$$

Note that in the matrix representation of (C.5), (C.6) and (C.7), the upper(left) entries denote the components with spinor indices $a = 1, \cdots, 4$, while the lower(right) ones denote $a = 5, \cdots, 8$. Also E_a, E_b and E_{b0} come from the overlapping condition of bosonic modes, fermionic non-zero modes and fermionic zero modes, respectively

$$E_a = \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty} \alpha^{i\dagger}_{(r)m}\tilde{N}^{(rs)}_{mn}\alpha^{i\dagger}_{(s)n}\right), \tag{C.8}$$

$$E_{b0} = \frac{1}{2^4}\prod_{a=1}^{4}(\sqrt{\alpha_{(1)}}b^{a\dagger}_{(1)} + \sqrt{\alpha_{(2)}}b^{a\dagger}_{(2)} + b^a_{(3)})\prod_{b=5}^{8}(\sqrt{\alpha_{(1)}}b^b_{(1)} + \sqrt{\alpha_{(2)}}b^b_{(2)} + b^{b\dagger}_{(3)}), \tag{C.9}$$

and the explicit expression of E_b is not necessary in our analysis. Finally the "ground" state $|0\rangle$ is related to the "vacuum" state with the lowest energy by:

$$|0\rangle = \prod_{a=5}^{8} b^{a\dagger}_{(1)}\prod_{b=5}^{8} b^{b\dagger}_{(2)}\prod_{c=1}^{4} b^{c\dagger}_{(3)}|\text{vac}\rangle. \tag{C.10}$$

Now we would like to calculate the supersymmetry charge matrix elements

$$Q_{n,m(s)} = \langle\text{vac}|\alpha^i_{n(3)}\alpha^i_{-n(3)}\alpha^k_{m(s)}\frac{1}{\sqrt{2}}(b^d_{m(s)} - ie(m)b^d_{-m(s)})|Q_{\dot{a}}\rangle, \tag{C.11}$$

where we assume the external states to be two bosonic impurity states and the intermediate states to be states with one bosonic and one fermionic impurity. The supersymmetry charge matrix elements under this assumption will be greatly simplified. We find the $(Y)^1$ and $(Y)^7$ terms in (C.3) vanish and the $(Y)^3$ and $(Y)^5$ terms reduce to (3.15).

The typical matrix element of the supersymmetry charge (C.11) is

$$\langle \mathrm{vac}|\alpha^i_{n(3)}\alpha^i_{-n(3)}\alpha^k_{m(s)}\frac{1}{\sqrt{2}}(b^d_{m(s)} - ie(m)b^d_{-m(s)})K^{\pm}Y^{\ell}E_aE_bE_{b0}|0\rangle, \tag{C.12}$$

where ℓ denotes the number of fermions in the expression (C.3). First of all, let us concentrate on the zero mode $b_{0(3)}$ operators. Since we only have $b_{0(3)}$ in E_{b0} and $|0\rangle$, all the $b_{0(3)}$ operators should cancel out to obtain a non-vanishing contribution. The only possibility is that $b^a_{0(3)}$ $(a = 1, \cdots, 4)$ in E_{b0} cancels those in $|0\rangle$ and we never use $b^b_{0(3)}$ $(b = 5, \cdots, 8)$ in E_{b0}. Using this fact, our typical matrix elements become

$$\langle \mathrm{vac}|\alpha^i_{n(3)}\alpha^i_{-n(3)}\alpha^k_{m(s)}\frac{1}{\sqrt{2}}(b^d_{m(s)} - ie(m)b^d_{-m(s)})K^{\pm}Y^{\ell}E_aE_{b0}\prod_{a=5}^{8} b^{a\dagger}_{0(1,2)}|\mathrm{vac}\rangle, \tag{C.13}$$

with $b^a_{0(1,2)}$ meaning $b^a_{0(1)}$ or $b^a_{0(2)}$. Next, let us concentrate on the zero mode $b_{0(1,2)}$ operators. In case of $\ell = 1$, we will not have enough annihilation operators to cancel all the leftover zero modes in $|0\rangle$. In case $\ell = 7$, we use four of $b(Y)^7$ to cancel the zero modes. But the rest must all be the creation operators and now we have too many of them. If $\ell = 3$, exactly four operators in $b(Y)^3$ are used to cancel the leftover in $|0\rangle$. If $\ell = 5$, four in $b(Y)^5$ are used to cancel. Since Y does not have both the creation operators and annihilation ones for the same operator, two of the Y's cannot cancel each other. Therefore we have to choose four operators in Y to cancel the creation operators in $|0\rangle$ and let the remaining Y's be cancelled by the b of the intermediate state.

For the $(Y)^3$ term, only the zero modes contribute:

$$\sim \bar{Y}_0(\tilde{F}^{\pm}_{(1)0}\tilde{N}^{(33)}_{n,-n} + \tilde{F}^{\pm}_{(3)n}\tilde{N}^{(13)}_{0,-n} + \tilde{F}^{\pm}_{(3)-n}\tilde{N}^{(13)}_{0,n}). \tag{C.14}$$

For the $(Y)^5$ term, besides the zero modes contribution, the non-zero modes also contribute as:

$$\sim \frac{\bar{Y}_{m(1)}}{\sqrt{2}}(\tilde{F}^{\pm}_{(1)m}\tilde{N}^{(33)}_{n,-n} + \tilde{F}^{\pm}_{(3)n}\tilde{N}^{(13)}_{m,-n} + \tilde{F}^{\pm}_{(3)-n}\tilde{N}^{(13)}_{m,n}). \tag{C.15}$$

Using the large μ behavior of various Neumann coefficients in Appendix A, we find that $\tilde{F}^{-}_{(3)}\tilde{N}^{(13)}$ gives the leading contribution. Besides, from the symmetry of the Neumann coefficients, we have

$$\tilde{F}^{-}_{(3)n}\tilde{N}^{(13)}_{0,-n} + \tilde{F}^{-}_{(3)-n}\tilde{N}^{(13)}_{0,n} \simeq 0. \tag{C.16}$$

Therefore the only relevant matrix element of the supersymmetry charge comes from $m \neq 0$.

For the analysis of the normalization of the contact term, let us be careful about the overall factor. Since

$$(\alpha_{(1)}\sqrt{\alpha_{(2)}}b^a_{0(2)}-\alpha_{(2)}\sqrt{\alpha_{(1)}}b^a_{0(1)})(\sqrt{\alpha_{(1)}}b^a_{0(1)}+\sqrt{\alpha_{(2)}}b^a_{0(2)})b^{a\dagger}_{0(1)}b^{a\dagger}_{0(2)}|\text{vac}\rangle = -\sqrt{\alpha_{(1)}\alpha_{(2)}}|\text{vac}\rangle, \tag{C.17}$$

for $a = 5, \cdots, 8$, from the cancellation of the fermionic zero modes we have an extra factor of

$$c_0 = \left(\frac{\sqrt{\alpha_{(1)}\alpha_{(2)}}}{2\bar{Y}_0}\right)^4. \tag{C.18}$$

Taking it into account, we find that the only non-trivial contribution is:

$$Q_{n,m(s)} = i\eta\frac{2cc_0}{\alpha^2}\sqrt{1+\mu\alpha k}u^i_{abc\dot{a}}\delta^{abcd}_{1234}\frac{\bar{Y}_{m(1)}}{\sqrt{2}}(\tilde{F}^-_{(3)-n}\tilde{N}^{(s3)}_{m,n}+\tilde{F}^-_{(3)n}\tilde{N}^{(s3)}_{m,-n}). \tag{C.19}$$

To fix the overall normalization, let us compare the supersymmetry charge matrix element with the Hamiltonian. If we restrict the external states to be purely bosonic ones, we also have the same fermionic zero mode factor c_0 in the Hamiltonian matrix element:

$$\begin{aligned}|H_3\rangle &= \frac{cc_0}{\alpha^2}(1+\mu\alpha k)(K^i_+-K^i_-)(K^j_++K^j_-)t^{ij}_{5678}E_a|\text{vac}\rangle \\ &= \frac{2cc_0}{\alpha^2}\left(-\frac{y(1-y)}{2}\right)\sum_{r=1}^{3}\sum_{n=-\infty}^{\infty}\frac{\omega_{n(r)}}{\alpha_{(r)}}\alpha^{i\dagger}_{n(r)}\alpha^i_{-n(r)}E_a|\text{vac}\rangle.\end{aligned} \tag{C.20}$$

In the final step, we have used the formula derived in [46]. Comparing our final expression with (3.6) whose normalization factor was determined in [9][8] by comparing the string field theory result with a gauge theory computation, we find that

$$\frac{2cc_0}{\alpha^2} = 1. \tag{C.21}$$

Appendix D. Formulas for calculating the contact term

The necessary summation and integration we need to calculate the contact term are the following ones:

$$\begin{aligned}\sum_{l=-\infty}^{\infty}\tilde{N}^{(13)}_{l,n}\tilde{N}^{(13)}_{l,m} &= \frac{(-1)^{m+n}\sin(n-m)\pi y}{\pi(n-m)}, \\ \sum_{l=-\infty}^{\infty}\tilde{N}^{(23)}_{l,n}\tilde{N}^{(23)}_{l,m} &= \frac{\sin(n-m)\pi(1-y)}{\pi(n-m)}.\end{aligned} \tag{D.1}$$

Also,

$$\int_0^1 dy(-1)^{m+n}\frac{\sin\pi my\sin\pi ny}{\pi^2}\Big\{(-1)^{l+n}\Big(\frac{\sin\pi(m-n)y}{\pi(m-n)}-\frac{\sin\pi(m+n)y}{\pi(m+n)}\Big)(1-y)$$
$$+\Big(\frac{\sin\pi(m-n)(1-y)}{\pi(m-n)}-\frac{\sin\pi(m+n)(1-y)}{\pi(m+n)}\Big)y\Big\}$$
$$=\frac{1}{4\pi^4(m-n)^2}+\frac{1}{4\pi^4(m+n)^2},$$
$$\int_0^1 dy\frac{\sin^2\pi ny}{\pi^2}\Big\{\Big(y-\frac{\sin 2\pi ny}{2\pi n}\Big)(1-y)+\Big((1-y)-\frac{\sin 2\pi n(1-y)}{2\pi n}\Big)y\Big\}$$
$$=\frac{1}{2\pi^2}\Big(\frac{1}{3}+\frac{5}{8\pi^2n^2}\Big). \tag{D.2}$$

Appendix E. $\tilde{\Gamma}^{(2)}$ computation

In this appendix, we explain the details of the computation of $\tilde{\Gamma}^{(2)}$ matrix elements for the operators with two impurities in the same direction, as discussed in section 3. The following identity will be useful throughout the computation:

$$C_{n,py}=C_{-n,-py}. \tag{E.1}$$

As in (2.4), $\tilde{\Gamma}^{(2)}$ is given by:

$$\tilde{\Gamma}^{(2)}=\Gamma^{(2)}-\frac{1}{2}\{G^{(2)},\Gamma^{(0)}\}-\frac{1}{2}\{G^{(1)},\Gamma^{(1)}\}+\frac{3}{8}\{G^{(1)2},\Gamma^{(0)}\}+\frac{1}{4}G^{(1)}\Gamma^{(0)}G^{(1)}. \tag{E.2}$$

Here we shall compute all the terms and show that $\tilde{\Gamma}^{(2)}$ reduces to (3.27).

Our strategy is to split each matrix element in (3.19) into a part proportional to δ_{ij} and a part coming from extra diagrams. More precisely, we have

$$\begin{aligned}\Gamma^{(1)}_{iin,jjqz}&=\delta_{ij}\big(\Gamma^{(1)}_{n,qz}+\Gamma^{(1)}_{-n,qz}\big)+\delta\Gamma^{(1)}_{n,qz},\\ \Gamma^{(1)}_{iin,jjz}&=\delta_{ij}\big(\Gamma^{(1)}_{n,z}+\Gamma^{(1)}_{-n,z}\big)+\delta\Gamma^{(1)}_{n,z},\\ \Gamma^{(2)}_{iin,jjm}&=\delta_{ij}\big(\Gamma^{(2)}_{n,m}+\Gamma^{(2)}_{n,-m}\big)+\delta\Gamma^{(2)}_{n,m},\end{aligned} \tag{E.3}$$

with

$$\begin{aligned}\delta\Gamma^{(1)}_{n,qz}&=-\frac{1}{2}\Gamma^{(1)}_{n,0z},\\ \delta\Gamma^{(1)}_{n,z}&=-\frac{1}{2}\Gamma^{(1)}_{n,z},\\ \delta\Gamma^{(2)}_{n,m}&=-\frac{1}{16\pi^2}\mathcal{D}^1_{n,m}.\end{aligned} \tag{E.4}$$

As a preliminary computation let us consider $(G^{(1)})^2$:

$$\begin{aligned}(G^{(1)})^2 &= J\int_0^1 dy\left(\sum_{p=1}^{\infty}\bigl(C_{n,py}+C_{n,-py}\bigr)\bigl(C_{py,m}+C_{-py,m}\bigr)+2C_{n,0y}C_{0y,m}\right)\\ &\qquad\qquad + J\int_0^{1/2} dy 2C_{n,y}2C_{y,m}\\ &= J\int_0^1 dy\sum_{p=-\infty}^{\infty}\Bigl(C_{n,py}C_{py,m}+C_{n,py}C_{py,-m}\Bigr)+2J\int_0^1 dyC_{n,y}C_{y,m}\\ &= \frac{1}{2}\bigl(M^1_{n,m}+M^1_{n,-m}\bigr).\end{aligned} \tag{E.5}$$

Here we have to be careful about the extra normalization factor $1/\sqrt{2}$ for zero modes as explained around (3.19). Note that originally in the first line we sum only over positive integers the product of two terms. One of the terms is the product of two contributions with opposite worldsheet momentum. But with the help of (E.1), we can rewrite these cross terms into the summation of two terms over all the integers, with still one of them carrying the reversed *external* worldsheet momentum as in the second equation in (E.5). Since one of two terms is identical to the one arising for operators with two impurities in different directions, we can perform the summation and integration easily and add the other term by reversing the external worldsheet momentum. This kind of mechanism happens everywhere, also in the computation of $\tilde{\Gamma}^{(2)}$. Therefore, the naive expectation of $\tilde{\Gamma}^{(2)}$ is obtained by adding a term with the external worldsheet momentum reversed:

$$\tilde{\Gamma}^{(2)}_{iin,jjm} = \delta_{ij}\bigl(\tilde{\Gamma}^{(2)}_{n,m}+\tilde{\Gamma}^{(2)}_{n,-m}\bigr) = \delta_{ij}\frac{1}{16\pi^2}\bigl(B_{n,m}+B_{n,-m}\bigr). \tag{E.6}$$

The only point we have to be careful with is whether (E.4) will give a non-trivial contribution.

Let us postpone the effect of (E.4) and concentrate on the dominant contribution to see whether the results have an additional contribution of reversing the worldsheet momentum, as compared to the case of operators with two impurities in different directions. Now it is quite trivial to calculate terms involving $\Gamma^{(0)}$ in (E.2) such as $\{(G^{(1)})^2,\Gamma^{(0)}\}$, $G^{(1)}\Gamma^{(0)}G^{(1)}$

and $\{G^{(2)}, \Gamma^{(0)}\}$. They are given by:

$$\begin{aligned}
\{(G^{(1)})^2, \Gamma^{(0)}\} &= \frac{n^2+m^2}{2}(M^1_{n,m} + M^1_{n,-m}), \\
G^{(1)}\Gamma^{(0)}G^{(1)} &= J\int_0^1 dy \sum_{p=0}^{\infty}(C_{n,py} + C_{n,-py})\frac{p^2}{y^2}(C_{py,m} + C_{-py,m}) \\
&= J\int_0^1 dy \sum_{p=-\infty}^{\infty}(C_{n,py}\frac{p^2}{y^2}C_{py,m} + C_{n,py}\frac{p^2}{y^2}C_{py,-m}), \\
\{G^{(2)}, \Gamma^{(0)}\} &= (n^2+m^2)(M^1_{n,m} + M^1_{n,-m}).
\end{aligned} \tag{E.7}$$

Let us turn to the term involving $\Gamma^{(1)}$ in (E.2), but ignoring the effect of (E.4). It is given by:

$$\begin{aligned}
&\{G^{(1)}, (\Gamma^{(1)} - \delta\Gamma^{(1)})\} \\
&= J\int_0^1 dy \sum_{p=1}^{\infty}(C_{n,py} + C_{n,-py})(\Gamma^1_{py,m} + \Gamma^1_{-py,m}) + \frac{1}{2}(C_{n,0y} + C_{n,0y})(\Gamma^1_{0y,m} + \Gamma^1_{0y,m}) \\
&\quad + J\int_0^1 dy \sum_{p=1}^{\infty}(\Gamma^1_{n,py} + \Gamma^1_{n,-py})(C_{py,m} + C_{-py,m}) + \frac{1}{2}(\Gamma^1_{n,0y} + \Gamma^1_{n,0y})(C_{0y,m} + C_{0y,m}) \\
&\quad + J\int_0^{1/2} dy(4C_{n,y}\Gamma^1_{y,m} + 4\Gamma^1_{n,y}C_{y,m}) \\
&= J\int_0^1 dy\left\{\sum_{p=-\infty}^{\infty}(C_{n,py}\Gamma^1_{py,m} + \Gamma^1_{n,py}C_{py,m}) + (C_{n,y}\Gamma^1_{y,m} + \Gamma^1_{n,y}C_{y,m})\right\} \\
&\quad + J\int_0^1 dy\left\{\sum_{p=-\infty}^{\infty}(C_{n,py}\Gamma^1_{py,-m} + \Gamma^1_{n,py}C_{py,-m}) + (C_{n,y}\Gamma^1_{y,-m} + \Gamma^1_{n,y}C_{y,-m})\right\}.
\end{aligned} \tag{E.8}$$

Also if we ignore the effect of (E.4), $\Gamma^{(2)}$ also has the same additional contribution, as seen in (E.3). As promised, all the results come paired with (n, m) and $(n, -m)$, where the first group of terms adds up to give the same result as for the case of two different impurities.

Now let us consider the contribution of $\delta\Gamma$'s to $\tilde{\Gamma}^{(2)}$

$$\delta\tilde{\Gamma}^{(2)} = \delta\Gamma^{(2)} - \frac{1}{2}\{G^{(1)}, \delta\Gamma^{(1)}\}, \tag{E.9}$$

that we have not taken into account so far. We can compute the second term as before:

$$
\begin{aligned}
&\{G^{(1)}, \delta\Gamma^{(1)}\}_{iin,jjm} \\
&= J\int_0^1 dy \left\{ \sum_{p=1}^{\infty} (C_{n,py} + C_{n,-py}) \left(-\frac{1}{2}\Gamma^{(1)}_{0y,m} \right) + C_{n,0y} \left(-\frac{1}{2}\Gamma^{(1)}_{0y,m} \right) \right\} \\
&\quad + J\int_0^1 dy \left\{ \sum_{p=1}^{\infty} \left(-\frac{1}{2}\Gamma^{(1)}_{n,0y} \right) (C_{py,m} + C_{-py,m}) + \left(-\frac{1}{2}\Gamma^{(1)}_{n,0y} \right) C_{0y,m} \right\} \\
&\quad + J\int_0^{\frac{1}{2}} dy (2C_{n,y}) \left(-\frac{1}{2}\Gamma^{(1)}_{y,m} \right) + \left(-\frac{1}{2}\Gamma^{(1)}_{n,y} \right) (2C_{y,n}) \\
&= -\frac{J}{2}\int_0^1 dy \sum_{p=-\infty}^{\infty} \left(C_{n,py}\Gamma^1_{0y,m} + \Gamma^1_{n,0y} C_{py,m} \right) - \frac{n^2+m^2}{2} J \int_0^1 dy C_{n,y} C_{y,m}.
\end{aligned}
\tag{E.10}
$$

Using the formula,

$$
J\int_0^1 dy \sum_{p=-\infty}^{\infty} C_{n,py}\Gamma^1_{0y,m} = \frac{1}{12\pi^2}\left(1 + \frac{3}{\pi^2 m^2}\right), \tag{E.11}
$$

and the summation formula in the appendix of [8], we obtain

$$
\{G^{(1)}, \delta\Gamma^{(1)}\}_{iin,jjm} = -\frac{1}{8\pi^2}\mathcal{D}^1_{n,m}, \tag{E.12}
$$

which precisely cancels $\delta\Gamma^{(2)}$:

$$
\delta\tilde{\Gamma}^{(2)} = 0. \tag{E.13}
$$

Therefore we find that (E.6) is exact. In section 3 and appendices B and C, we saw that this result is correctly reproduced from the contact term calculation in string field theory.

Appendix F. Anomalous dimension of the singlet operators

In this appendix we shall calculate the anomalous dimension of the operator with two impurities in the same direction, using the perturbation theory. This calculation has essentially been done in [15] by diagonalizing the matrix of two-point functions in the BMN basis. Here, we perform the calculation using the string field theory basis and it serves as a consistency check of the evaluation of $\tilde{\Gamma}^{(2)}$ in the previous appendix.

In perturbation theory the eigenvalue at $O(g_2^2)$ is given by:

$$
\Delta^{(2)} = J\int_0^1 dy \sum_{p=0}^{\infty}\sum_{j=1}^{4} \frac{(\tilde{\Gamma}^{(1)}_{ii;n,jj;py})^2}{n^2 - p^2/y^2} + J\int_0^{1/2} dy \sum_{j=1}^{4} \frac{(\tilde{\Gamma}^{(1)}_{ii;n,jj;y})^2}{n^2} + \tilde{\Gamma}^{(2)}_{ii;n,ii;n}. \tag{F.1}
$$

Using the following relations

$$\tilde{\Gamma}^{(1)}_{n,py} = \frac{1}{2}\Gamma^{(1)}_{n,0y},$$
$$\tilde{\Gamma}^{(1)}_{n,y} = \frac{1}{2}\Gamma^{(1)}_{n,y}, \tag{F.2}$$

(F.1) can be rewritten as:

$$\Delta^{(2)} = \frac{J}{2}\int_0^1 dy \sum_{p=-\infty}^{\infty} \frac{(\Gamma^{(1)}_{n,0y})^2}{n^2 - p^2/y^2} + \frac{J}{2}\int_0^1 dy \frac{(\Gamma^{(1)}_{n,y})^2}{n^2} + \tilde{\Gamma}^{(2)}_{ii;n,ii;n}. \tag{F.3}$$

Now let us proceed to evaluate each term. The first term is

$$\frac{1}{2\pi^4}\int_0^1 dy \frac{1-y}{y}\sin^4(\pi n y) \sum_{p=-\infty}^{\infty} \frac{1}{n^2 - p^2/y^2} = \frac{3}{64\pi^4 n^2}, \tag{F.4}$$

and the second term is simply reduced to the integration of $(C_{n,y})^2$, whose result can be found in [8]:

$$\frac{n^2}{2} J \int_0^1 dy C^2_{n,y} = \frac{3}{16\pi^4 n^2}. \tag{F.5}$$

Consequently, the anomalous dimension of the singlet operator is

$$\Delta^{(2)} = \frac{3}{64\pi^4 n^2} + \frac{3}{16\pi^4 n^2} + \frac{1}{16\pi^2}\big(B_{n,n} + B_{n,-n}\big) = \frac{1}{16\pi^2}\left(\frac{1}{3} + \frac{35}{8\pi^2 n^2}\right), \tag{F.6}$$

which as explained in [15][16] is the same as the operators transforming in the **6** and **9** representations of $SO(4)$.

Appendix G. BMN operators in complex field notation

In the main text, we have used real scalar field notation to define BMN operators with arbitrary combination of impurities. In this case, we have four kinds of scalar impurities $\phi_1, \phi_2, \phi_3, \phi_4$ which can be inserted, and a subtlety arises when two identical impurities collide. In this appendix, we study the same problem in the complex scalar field formulation. In this formulation, there are also four kinds of impurities $\Phi, \bar{\Phi}, \Psi, \bar{\Psi}$. First, we want to see if BMN operators with anti-holomorphic insertions are well defined in the BMN limit. For example, let us consider Φ and $\bar{\Psi}$ insertions:

$$\mathcal{O}^J_{\Phi\bar{\Psi},n} = \frac{1}{\sqrt{JN^{J+2}}}\sum_{l=0}^{J} e^{2\pi i l n/J}\mathrm{Tr}\left(\Phi Z^l \bar{\Psi} Z^{J-l}\right). \tag{G.1}$$

From the original Lagrangian of $\mathcal{N} = 4$ SYM theory, it is easy to see that there is a symmetry which maps ϕ_4 to $-\phi_4$, thereby transforming Ψ to $\bar{\Psi}$ without changing Z and Φ. From the ten-dimensional $\mathcal{N} = 1$ SYM viewpoint, it is just the reflection along one of the internal directions. In terms of an $\mathcal{N} = 1$ superfield formulation of $\mathcal{N} = 4$ SYM, it is equivalent to treating $Z, \Phi, \bar{\Psi}$ as chiral superfields instead of Z, Φ, Ψ. (The original D-term potential and F-term potential should regroup to give the same form of D-term and F-term potential in terms of $Z, \Phi, \bar{\Psi}$.) Therefore, the Feynman diagram computation is identical to that for BMN operators with Φ and Ψ insertions, as it should be because in terms of the real scalar representation the four impurities are equivalent as far as same impurities do not collide. Hence, we conclude that the four complex impurities are equivalent in the dilute gas approximation.

Now let us think about the subtlety arising when two impurities collide. In the real scalar representation, only when two same impurities collide we had to add an extra term with $\bar{Z}$ insertion. In the complex scalar representation, this extra term is necessary only when Φ and $\bar{\Phi}$ collide or Ψ and $\bar{\Psi}$ collide. This can be understood from the action of R-symmetry generators on the BMN operators.(See also [35].) Let us denote the R-symmetry generator of the rotation on ϕ_i-ϕ_j plane by $R_{ij} = -R_{ji}$. More precisely,

$$R_{ij} \cdot \phi_j = \phi_i, \qquad R_{ij} \cdot \phi_i = -\phi_j. \tag{G.2}$$

Then define

$$R_{\Phi Z} = \frac{1}{2}\left(R_{15} + R_{26} + iR_{25} - iR_{16}\right), \qquad R_{\bar{\Phi} Z} = \frac{1}{2}\left(R_{15} - R_{26} - iR_{25} - iR_{16}\right). \tag{G.3}$$

Their actions are given as

$$\begin{aligned} &R_{\Phi Z} \cdot Z = \Phi, \qquad R_{\Phi Z} \cdot \bar{\Phi} = -\bar{Z}, \qquad R_{\Phi Z} \cdot \bar{Z} = R_{\Phi Z} \cdot \Phi = 0, \\ &R_{\bar{\Phi} Z} \cdot Z = \bar{\Phi}, \qquad R_{\bar{\Phi} Z} \cdot \Phi = -\bar{Z}, \qquad R_{\bar{\Phi} Z} \cdot \bar{Z} = R_{\bar{\Phi} Z} \cdot \bar{\Phi} = 0, \end{aligned} \tag{G.4}$$

and likewise for $R_{\Psi Z}$ and $R_{\bar{\Psi} Z}$. BPS BMN operators can be obtained by acting these generators successively on the vacuum operator $\mathrm{Tr}(Z^J)$. For example, if we want to insert Φ and Ψ, we act with $R_{\Phi Z}$ and $R_{\Psi Z}$ on $\mathrm{Tr}(Z^{J+2})$,

$$R_{\Psi Z} \cdot \left(R_{\Phi Z} \cdot \mathrm{Tr}(Z^{J+2})\right) = R_{\Psi Z} \cdot \left(\sum_{l=0}^{J+1} \mathrm{Tr}(Z^l \Phi Z^{J+1-l})\right) = (J+2) \sum_{l=0}^{J} \mathrm{Tr}(\Phi Z^l \Psi Z^{J-l}). \tag{G.5}$$

Since $R_{\Psi Z} \cdot \Phi = 0$, we don't have any extra term arising when $R_{\Psi Z}$ acts on Φ. It is also the case when we insert two Φ's because $R_{\Phi Z} \cdot \Phi = 0$. Now let us consider Φ and $\bar{\Phi}$ insertions.

$$\begin{aligned} R_{\bar{\Phi} Z} \cdot \left(R_{\Phi Z} \cdot \mathrm{Tr}(Z^{J+2})\right) &= R_{\bar{\Phi} Z} \cdot \left(\sum_{l=0}^{J+1} \mathrm{Tr}(Z^l \Phi Z^{J+1-l})\right) \\ &= (J+2)\left(\sum_{l=0}^{J} \mathrm{Tr}(\Phi Z^l \bar{\Phi} Z^{J-l}) - \mathrm{Tr}(\bar{Z} Z^{J+1})\right). \end{aligned} \tag{G.6}$$

The $\bar{Z}$ term arises when $R_{\bar{\Phi} Z}$ acts on Φ, in other words, when Φ and $\bar{\Phi}$ "collide". We conclude that only when holomorphic and antiholomorphic insertions of the same kind collide, we need to add an extra $\bar{Z}$ term. From this consideration, we can also learn that no extra term is necessary when $\bar{Z}$ collides with the four impurities because all the four generators annihilate $\bar{Z}$. For example, when $R_{\Phi Z}$ acts on $\mathrm{Tr}(\bar{Z} Z^{J+1})$,

$$R_{\Phi Z} \cdot \mathrm{Tr}(\bar{Z} Z^{J+1}) = \sum_{l=0}^{J} \mathrm{Tr}(\bar{Z} Z^l \Phi Z^{J-l}). \tag{G.7}$$

This implies that we don't have to worry about collision of more than two impurities. In general, we have only to take care of holomorphic and antiholomorphic impurities of the same kind pairwise.

Appendix H. Off-shell representation of BMN operators

In this appendix, we carefully define "on-shell" and "off-shell" representations of BMN operators which are introduced in the main text. (See also [14].) Here by shell we mean the level matching condition shell, which states that the sum of all worldsheet momentum vanishes. In the on-shell representation, we fix the position of one scalar impurity and sum over positions of the rest of the impurities. To explain more explicitly, let us consider the case of three impurities. In this case, we have

$$\mathcal{O}^J_{\mathrm{on}} = \sum_{0 \le l_2, l_3 \le J} \mathrm{Tr}\left(\phi_{d_1} Z \cdots Z \phi_{d_2} Z \cdots Z \phi_{d_3} Z \cdots Z\right) s_2^{l_2} s_3^{l_3}, \tag{H.1}$$

where $d_i \in \{1, 2, 3, 4\}$ is the direction of the i-th impurity, l_i is the number[24] of Z in front of ϕ_{d_i} and $s_i = e^{2\pi i n_i / J}$ is the phase assigned to ϕ_{d_i}. This definition gets ambiguous when two impurities sit next to each other. Therefore, we need a rigorous definition:

$$\mathcal{O}^J_{\mathrm{on}} = \mathcal{O}^J_{\mathrm{on,c}} + \mathcal{O}^J_{\mathrm{on,a}}, \tag{H.2}$$

[24] In [35], l_i is argued to include the number of other impurities in front of it, but the difference is only subleading in $1/J$ and inconsequential throughout this paper.

with

$$\begin{aligned}\mathcal{O}^J_{\text{on,c}} &= \sum_{0\le a_2\le a_3\le J} \text{Tr}\Big(\phi_{d_1} Z^{a_2}\phi_{d_2} Z^{a_3-a_2}\phi_{d_3} Z^{J-a_3}\Big) s_2^{a_2} s_3^{a_3},\\ \mathcal{O}^J_{\text{on,a}} &= \sum_{0\le a_3\le a_2\le J} \text{Tr}\Big(\phi_{d_1} Z^{a_3}\phi_{d_3} Z^{a_2-a_3}\phi_{d_2} Z^{J-a_2}\Big) s_2^{a_2} s_3^{a_3}.\end{aligned} \tag{H.3}$$

To normalize this operator canonically, let us compute its free two-point function:

$$\langle \bar{\mathcal{O}}^J_{\text{on}} \mathcal{O}^J_{\text{on}}\rangle = \langle \bar{\mathcal{O}}^J_{\text{on,c}} \mathcal{O}^J_{\text{on,c}}\rangle + \langle \bar{\mathcal{O}}^J_{\text{on,a}} \mathcal{O}^J_{\text{on,a}}\rangle = (J+1)(J+2)N^{J+3}. \tag{H.4}$$

Here we have counted the number of pairs (a_2, a_3) such that $0 \le a_2 \le a_3 \le J$, which is $\binom{J+2}{2}$. Hence, in the BMN limit, the correct normalization is:

$$\mathcal{O}^J_{\text{BMN}} = \frac{1}{J\sqrt{N^{J+3}}}\mathcal{O}^J_{\text{on}}. \tag{H.5}$$

Now let us move on to the off-shell representation. In the off-shell representation, we do not fix the position of any scalar impurity and treat them on equal footing by summing over all possible positions of all impurities. For our present case of 3 impurities, we define

$$\mathcal{O}^J_{\text{off}} = \sum_{0\le l_1,l_2,l_3\le J} \text{Tr}\left(Z\cdots Z\phi_{d_1} Z\cdots Z\phi_{d_2} Z\cdots Z\phi_{d_3} Z\cdots Z\right) s_1^{l_1} s_2^{l_2} s_3^{l_3}, \tag{H.6}$$

where l_i is defined in the same way as above. Again, a rigorous definition is given by

$$\mathcal{O}^J_{\text{off}} = \mathcal{O}^J_{\text{off}}(1,2,3) + \mathcal{O}^J_{\text{off}}(2,3,1) + \mathcal{O}^J_{\text{off}}(3,1,2) + \mathcal{O}^J_{\text{off}}(1,3,2) + \mathcal{O}^J_{\text{off}}(3,2,1) + \mathcal{O}^J_{\text{off}}(2,1,3), \tag{H.7}$$

with

$$\begin{aligned}\mathcal{O}^J_{\text{off}}(1,2,3) &= \sum_{0\le a_1\le a_2\le a_3\le J} \text{Tr}\left(Z^{a_1}\phi_{d_1} Z^{a_2-a_1}\phi_{d_2} Z^{a_3-a_2}\phi_{d_3} Z^{J-a_3}\right) s_1^{a_1} s_2^{a_2} s_3^{a_3},\\ \mathcal{O}^J_{\text{off}}(2,3,1) &= \sum_{0\le a_2\le a_3\le a_1\le J} \text{Tr}\left(Z^{a_2}\phi_{d_2} Z^{a_3-a_2}\phi_{d_3} Z^{a_1-a_3}\phi_{d_1} Z^{J-a_1}\right) s_1^{a_1} s_2^{a_2} s_3^{a_3},\\ &\vdots\end{aligned} \tag{H.8}$$

where the other operators are defined likewise. Now the claim is that $\mathcal{O}^J_{\text{off}}$ is non-vanishing if and only if $n_1+n_2+n_3=0$:

$$\mathcal{O}^J_{\text{off}} \neq 0 \Longleftrightarrow n_1+n_2+n_3=0. \tag{H.9}$$

Note that this condition is exactly the level-matching condition in string field theory. Furthermore, if this condition holds, we have

$$\begin{aligned}\mathcal{O}^J_{\text{off}}(1,2,3)+\mathcal{O}^J_{\text{off}}(2,3,1)+\mathcal{O}^J_{\text{off}}(3,1,2)&=J\mathcal{O}^J_{\text{on,c}},\\ \mathcal{O}^J_{\text{off}}(1,3,2)+\mathcal{O}^J_{\text{off}}(3,2,1)+\mathcal{O}^J_{\text{off}}(2,1,3)&=J\mathcal{O}^J_{\text{on,a}},\end{aligned} \tag{H.10}$$

and the off-shell representation (H.7) is reduced to the on-shell one (H.2):

$$\mathcal{O}^J_{\text{off}}=J\mathcal{O}^J_{\text{on}}. \tag{H.11}$$

This explains the terminology of "on-shell/off-shell" representation. Consequently, the correct normalization of the off-shell operator is

$$\mathcal{O}^J_{\text{BMN}}=\frac{1}{\sqrt{J}\sqrt{J^3}\sqrt{N^{J+3}}}\mathcal{O}^J_{\text{off}}. \tag{H.12}$$

This argument can be immediately generalized to n impurities assuming all of them are different. The on-shell operator is

$$\mathcal{O}^J_{\text{on}}=\sum_{0\le l_2,\cdots,l_n\le J}\text{Tr}\left(\phi_{d_1}Z\cdots Z\phi_{d_2}Z\cdots\cdots Z\phi_{d_n}Z\cdots Z\right)\prod_{i=2}^{n}s_i^{l_i}, \tag{H.13}$$

with l_i being the number of Z's in front of ϕ_i as before. Or more rigorously the definition of it is given as the sum of $(n-1)!$ operators corresponding to permutations after fixing the position of one impurity:

$$\mathcal{O}^J_{\text{on}}=\sum_{\sigma\in\text{Perm}\{2,\cdots,n\}}\mathcal{O}^J_{\text{on},\sigma}, \tag{H.14}$$

with

$$\mathcal{O}^J_{\text{on},\sigma}=\sum_{0\le a_{\sigma(2)}\le\cdots\le a_{\sigma(n)}\le J}\text{Tr}\left(\phi_{d_1}Z^{a_{\sigma(2)}}\phi_{d_{\sigma(2)}}Z^{a_{\sigma(3)}-a_{\sigma(2)}}\phi_{d_{\sigma(3)}}\cdots\phi_{d_{\sigma(n)}}Z^{J-a_{\sigma(n)}}\right)\prod_{i=2}^{n}s_i^{a_i}. \tag{H.15}$$

Each $\mathcal{O}^{J,n}_{on,\sigma}$ is composed of $\begin{pmatrix}J+n-1\\ n-1\end{pmatrix}\simeq\dfrac{J^{n-1}}{(n-1)!}$ terms, where the combinatoric number comes from the number of $(n-1)$-tuple $(a_2,a_3,\cdots,a_n)$ satisfying $0\le a_{\sigma(2)}\le\cdots\le a_{\sigma(n)}\le J$. Hence, the normalization is

$$\mathcal{O}^J_{\text{BMN}}=\frac{1}{\sqrt{J^{n-1}}\sqrt{N^{J+n}}}\mathcal{O}^J_{\text{on}}. \tag{H.16}$$

Similarly, the off-shell operator with n impurities is

$$\mathcal{O}^J_{\rm off} = \sum_{0\le l_1,\cdots,l_n\le J} {\rm Tr}\,(Z\cdots Z\phi_{d_1}Z\cdots Z\phi_{d_2}Z\cdots\cdots Z\phi_{d_n}Z\cdots Z)\prod_{i=1}^{n} s_i^{l_i}, \tag{H.17}$$

with a rigorous definition given by a sum over $n!$ terms. As in 3-impurity case, we have

$$\mathcal{O}^J_{\rm off} = J\mathcal{O}^J_{\rm on}, \tag{H.18}$$

if and only if the level-matching (on-shell) condition holds for the off-shell operator. Hence the normalization for the off-shell operator is

$$\mathcal{O}^J_{\rm BMN} = \frac{1}{\sqrt{J}\sqrt{J^n}\sqrt{N^{J+n}}}\mathcal{O}^J_{\rm off}. \tag{H.19}$$

Here we can think of each impurity as carrying a normalization factor $1/\sqrt{J}$, since we sum over J possible positions for each impurity. The leftover factor $1/\sqrt{J}$ is the original normalization of the vacuum operator and it originates in the cyclic property of Tr.

So far, we have defined on-shell and off-shell operators assuming that all impurities are distinct. However, we have to deal with same impurities eventually since there are only 4 directions. When two impurities, say ϕ_{d_1} and ϕ_{d_2}, are the same, we have to insert $-\bar{Z}$ when they collide as discussed in Appendix G. Then the correct definition is in the off-shell representation,

$$\begin{aligned}\mathcal{O}^J_{\rm off} = &\sum_{0\le l_1,l_2,l_3,\cdots,l_n\le J} {\rm Tr}\,(Z\cdots Z\phi_{d_1}Z\cdots Z\phi_{d_2}Z\cdots Z\phi_{d_3}Z\cdots\cdots Z\phi_{d_n}Z\cdots Z)\prod_{i=1}^{n} s_i^{l_i} \\ &- \sum_{0\le l_{(1,2)},l_3,\cdots,l_n\le J+1} {\rm Tr}\left(Z\cdots Z\bar{Z}Z\cdots Z\phi_{d_3}Z\cdots\cdots Z\phi_{d_n}Z\cdots Z\right)(s_1s_2)^{l_{(1,2)}}\prod_{i=3}^{n} s_i^{l_i},\end{aligned} \tag{H.20}$$

where $l_{(1,2)}$ is the number of Z's in front of $\bar{Z}$ arising when ϕ_{d_1} and ϕ_{d_2} collide. Now we have to do this modification whenever we have a pair (i,j) such that $d_i = d_j$. However, as argued in Appendix G, we do not have to worry about collision of more than two impurities. The normalization is not changed since the number of $\bar{Z}$ terms is subleading in $1/J$ compared with the original terms because $\bar{Z}$ terms arise only when two impurities collide.

As an example, let us consider a BMN operator with 4 same impurities, i.e. $d_1 = d_2 = d_3 = d_4$. In this case the BMN operator should be modified by $-\bar{Z}$ as

$$\begin{aligned}
\mathcal{O}^J_{\text{off}} = & \sum_{0\le l_1,l_2,l_3,l_4\le J} \mathrm{Tr}\left(Z\cdots Z\phi_{d_1}Z\cdots Z\phi_{d_2}Z\cdots Z\phi_{d_3}Z\cdots Z\phi_{d_4}Z\cdots Z\right)\prod_{i=1}^{4} s_i^{l_i} \\
- & \sum_{0\le l_{(1,2)},l_3,l_4\le J+1} \mathrm{Tr}\left(Z\cdots Z\bar{Z}Z\cdots Z\phi_{d_3}Z\cdots Z\phi_{d_4}Z\cdots Z\right)(s_1s_2)^{l_{(1,2)}}s_3^{l_3}s_4^{l_4} \\
- & \sum_{0\le l_{(1,3)},l_2,l_4\le J+1} \mathrm{Tr}\left(Z\cdots Z\bar{Z}Z\cdots Z\phi_{d_2}Z\cdots Z\phi_{d_4}Z\cdots Z\right)(s_1s_3)^{l_{(1,3)}}s_2^{l_2}s_4^{l_4} \\
& \vdots \\
+ & \sum_{0\le l_{(1,2)},l_{(3,4)}\le J+2} \mathrm{Tr}\left(Z\cdots Z\bar{Z}Z\cdots Z\bar{Z}Z\cdots Z\right)(s_1s_2)^{l_{(1,2)}}(s_3s_4)^{l_{(3,4)}} \\
& \vdots
\end{aligned} \tag{H.21}$$

Appendix I. The $\mathcal{O}(g_2)$ coupling an p-th string state to an $p+1$-th string state

In a recent paper Gursoy [33] has analyzed the two-point function of multi-trace BMN operators with two different impurities. The class of operators he considers are:

$$\mathcal{T}^{J,y_1,y_2,\ldots,y_p}_{ij,n} =: \mathcal{O}^{y_1\cdot J}_n\cdot\mathcal{O}^{y_2\cdot J}\cdots\mathcal{O}^{y_p\cdot J}:\delta_{y_1+\ldots+y_p,1}. \tag{I.1}$$

The $\mathcal{O}(g_2)$ result for the mixing of the p-th trace with the $p+1$ trace BMN operator is given by [33]

$$\begin{aligned}
G^{p,p+1(1)}_{ny_1\ldots y_p;mz_1\ldots z_p} = y_1^{3/2}C_{n,mz_1/y_1}\sum_P \delta_{y_2,z_{P(2)}}\cdots\delta_{y_p,z_{P(p)}}\delta_{y_1,z_{P(p)}}\delta_{y_1,z_1+z_{P(i+1)}}+ \\
\frac{1}{J}\delta_{n,m}\delta_{y_1,z_1}\sum_{P,P'}\delta_{y_{P(2)},z_{P'(2)}}\cdots\delta_{y_{P(p-1)},z_{P'(p-1)}}\delta_{y_{P(p)},z_{P'(p)}+z_{P'(p+1)}}.
\end{aligned} \tag{I.2}$$

The contribution in the first line is due to contractions in which the two impurities in the p-trace operator contract with the two impurities and any vacuum operator in the $p+1$-trace operator. Therefore, the quantity $C_{n,mz_1/y_1}$ is the mixing between a single trace and double trace two-impurity BMN operator. The contribution in the second line comes from Wick contractions where the operators with the impurities just connect among themselves and the vacuum operators in the p-th trace BMN operator contract with the vacuum operator in the $p+1$-th trace BMN operator.

The contribution to the the matrix of anomalous dimensions is given by [33]

$$\Gamma^{p,p+1(1)}_{ny_1\dots y_p;mz_1\dots z_p} = \left(\frac{n^2}{y_1^2} + \frac{m^2}{z_1^2} - \frac{nm}{y_1 z_1}\right) G^{p,p+1(1)}_{ny_1\dots y_p;mz_1\dots z_p}. \tag{I.3}$$

Using the holographic proposal we can calculate these matrix elements in the orthonormal basis:

$$\begin{aligned}\tilde{\Gamma}^{p,p+1(1)}_{ny_1\dots y_p;mz_1\dots z_p} &= \Gamma^{p,p+1(1)}_{ny_1\dots y_p;mz_1\dots z_p} - \frac{1}{2}\left\{G^{p,p+1(1)}_{ny_1\dots y_p;mz_1\dots z_p}, \Gamma^{p,p+1(0)}_{ny_1\dots y_p;mz_1\dots z_p}\right\} \\ &= \frac{1}{2}\left(\frac{n}{y_1} - \frac{m}{z_1}\right)^2 G^{p,p+1(1)}_{ny_1\dots y_p;mz_1\dots z_p}.\end{aligned} \tag{I.4}$$

We note that in the orthonormal basis that the second term in (I.2) does not contribute due to the δ function constraints and the prefactor in (I.4). Therefore, the final answer can be written as:

$$\tilde{\Gamma}^{p,p+1(1)}_{ny_1\dots y_p;mz_1\dots z_p} = \frac{1}{\sqrt{y_1}}\tilde{\Gamma}_{n,mz_1/y_1} \times \sum_P \delta_{y_2,z_{P(2)}} \dots \delta_{y_p,z_{P(p)}} \delta_{y_1,z_{P(p)}} \delta_{y_1,z_1+z_{P(i+1)}}. \tag{I.5}$$

We now perform the relevant string field theory calculation. The string states dual to the BMN operators (I.1) are given by

$$|n, y_1, y_2, \dots, y_p\rangle = \alpha^i_n \alpha^j_{-n} |\text{vac}, y_1\rangle \otimes |\text{vac}, y_2\rangle \otimes \dots \otimes |\text{vac}, y_p\rangle \; \delta_{y_1+\dots+y_p,1}. \tag{I.6}$$

It follows from the expression for the cubic Hamiltonian vertex (3.6)(3.7) that any contraction coupling only vacua is zero. The only possible non-zero contributions are those in which the string carrying the two impurities contracts with the string carrying two-impurities and a vacuum string state. Therefore, the matrix elements of the unitly normalized states are:

$$\begin{aligned}&\frac{1}{\mu}\langle n, y_1 \dots y_p | H_3 | m, z_1 \dots z_p\rangle \\ &= \frac{1}{\mu}\langle n, y_1 | H_3 | m, z_1\rangle \times \sum_P \delta_{y_2,z_{P(2)}} \dots \delta_{y_p,z_{P(p)}} \delta_{y_1,z_{P(p)}} \delta_{y_1,z_1+z_{P(i+1)}} \\ &= \frac{1}{\sqrt{y_1}}\tilde{\Gamma}^{(1)}_{n,mz_1/y_1} \times \sum_P \delta_{y_2,z_{P(2)}} \dots \delta_{y_p,z_{P(p)}} \delta_{y_1,z_{P(p)}} \delta_{y_1,z_1+z_{P(i+1)}},\end{aligned} \tag{I.7}$$

which match the gauge theory computation.

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